

# FILTERS IN THE PARTITION LATTICE

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**ABSTRACT.** Given a filter  $\Delta$  in the poset of compositions of  $n$ , we form the filter  $\Pi_\Delta^*$  in the partition lattice. We determine all the reduced homology groups of the order complex of  $\Pi_\Delta^*$  as  $\mathfrak{S}_{n-1}$ -modules in terms of the reduced homology groups of the simplicial complex  $\Delta$  and in terms of Specht modules of border shapes. We also obtain the homotopy type of this order complex. These results generalize work of Calderbank–Hanlon–Robinson and Wachs on the  $d$ -divisible partition lattice. Our main theorem applies to a plethora of examples, including filters associated to integer knapsack partitions and filters generated by all partitions having block sizes  $a$  or  $b$ . We also obtain the reduced homology groups of the filter generated by all partitions having block sizes belonging to the arithmetic progression  $a, a + d, \dots, a + (a - 1) \cdot d$ , extending work of Browdy.

## 1. INTRODUCTION

In his physics dissertation Sylvester [19] considered the even partition lattice, that is, the poset of all set partitions where the blocks have even size. He computed the Möbius function of this lattice and showed that it equals, up to a sign, the tangent number. Stanley then introduced the  $d$ -divisible partition lattice. This is the collection of all set partitions with blocks having size divisible by  $d$ , denoted by  $\Pi_n^d$ . He showed that the Möbius function is, up to a sign, the number of permutations in the symmetric group  $\mathfrak{S}_{n-1}$  with descent set  $\{d, 2d, \dots, n - d\}$ ; see [14].

Calderbank, Hanlon and Robinson [5] continued this work by studying the top homology group of the order complex  $\Delta(\Pi_n^d - \{\hat{1}\})$  and gave an explicit description of the  $\mathfrak{S}_{n-1}$ -action on this homology group in terms of a Specht module. However, they were unable to obtain the other homology groups and asked Wachs if it was possible that the complex  $\Delta(\Pi_n^d - \{\hat{1}\})$  was shellable, which would imply that the other homology groups are trivial. Wachs [20] proved that this was indeed the case by showing that the poset  $\Pi_n^d \cup \{\hat{0}\}$  is *EL*-shellable, and thus the homotopy type of the complex  $\Delta(\Pi_n^d - \{\hat{1}\})$  is a wedge of spheres of the same dimension. Additionally, Wachs gave a different proof for the  $\mathfrak{S}_{n-1}$ -action on the top homology of  $\Pi_n^d$ , as well as matrices for the action of  $\mathfrak{S}_n$  on this homology.

Ehrenborg and Jung [7] further generalized the  $d$ -divisible partition lattice by defining a subposet  $\Pi_{\vec{c}}^*$  of the partition lattice for a composition  $\vec{c}$  of  $n$ . The subposet reduces to the  $d$ -divisible partition lattice when the composition  $\vec{c}$  is given by  $\vec{c} = (d, d, \dots, d)$ . Their work consists of three main results. First, they showed that the Möbius function of  $\Pi_{\vec{c}}^* \cup \{\hat{0}\}$  equals, up to a given sign, the number of permutations in  $\mathfrak{S}_n$  ending with the element  $n$  having descent composition  $\vec{c}$ . Second, they showed that the order complex  $\Delta(\Pi_{\vec{c}}^* - \{\hat{1}\})$  is homotopy equivalent to a wedge of spheres of the same dimension. Lastly, they proved that the action of  $\mathfrak{S}_{n-1}$  on the top homology group of  $\Delta(\Pi_{\vec{c}}^* - \{\hat{1}\})$  is given by the Specht module corresponding to the composition  $\vec{c} - 1$ .

In the current paper we continue this research program by considering a more general class of filters in the partition lattice. Let  $\Delta$  be a filter in the poset of compositions. Since the poset of compositions is isomorphic to a Boolean algebra, the filter  $\Delta$  under the reverse order is a lower order ideal and hence can be viewed as the face poset of a simplicial complex. We define the associated

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filter  $\Pi_\Delta^*$  in the partition lattice. This extends the definition of  $\Pi_{\vec{c}}^*$ . In fact, when  $\Delta$  is a simplex generated by the composition  $\vec{c}$  the two definitions agree.

Our main result is that we can determine all the reduced homology groups of the order complex  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  in terms of the reduced homology groups of links in  $\Delta$  and of Specht modules of border shapes; see Theorem 11.6. The proof proceeds by showing that if the result holds for the two complexes  $\Delta, \Gamma$  and also for their intersection  $\Delta \cap \Gamma$ , then it holds for their union  $\Delta \cup \Gamma$ . Furthermore, the proof relies on Mayer–Vietoris sequences to construct the isomorphism of Theorem 6.4. As our main tool, we use Quillen’s fiber lemma to translate topological data from the filter  $Q_\Delta^*$  to the filter  $\Pi_\Delta^*$ .

We also present a second proof of our main result, Theorem 6.4, using an equivariant poset fiber theorem of Björner, Wachs and Welker [3]. Even though this approach is concise, it does not yield an explicit construction of the isomorphism of Theorem 6.4. In particular, our hands on approach using Mayer–Vietoris sequences reveals how the homology groups of  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  are changing as the complex  $\Delta$  is built up. Once again, the Ehrenborg–Jung result on  $Q_{\vec{c}}^*$  is needed to apply the poset fiber theorem.

Our main result yields explicit expressions for the reduced homology groups of the complex  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$ , most notably when  $\Delta$  is homeomorphic to a ball or to a sphere. The same holds when  $\Delta$  is a shellable complex. We are able to describe the homotopy type of the order complex  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  using the homotopy fiber theorem of [3]. Again, when  $\Delta$  is homeomorphic to a ball or to a sphere, we obtain that  $\Pi_\Delta^*$  is a wedge of spheres. We are also able to lift discrete Morse matchings from  $\Delta$  and its links to form a discrete Morse matching on the filter of ordered set partitions  $Q_\Delta^*$ .

In Sections 15 through 17 we give a plethora of examples of our results. We consider the case when the complex  $\Delta$  is generated by a knapsack partition to obtain a previous result of Ehrenborg and Jung. In Section 16 we study the case when  $\Lambda$  is a semigroup of positive integers and we consider the filter of partitions whose block sizes belong to the semigroup  $\Lambda$ . When  $\Lambda$  is generated by the arithmetic progression  $a, a + d, a + 2d, \dots$  we are able to describe the reduced homology groups of the associated filter in the partition lattice. The particular case when  $d$  divides  $a$  was studied by Browdy [4], where the filter  $\Lambda$  consists of partitions whose block sizes are divisible by  $d$  and are greater than or equal to  $a$ . Finally, in Section 17 we study the filter corresponding to the semigroup generated by two relative prime integers. Here we are able to give explicit results for the top and bottom reduced homology groups.

Other previous work in this area is due to Björner and Wachs [2]. Additionally, Sundaram studied the subposet of the partition lattice defined by a set of forbidden block sizes using plethysm and the Hopf trace formula; see [17, 18].

We end the paper by posing questions for further study.

## 2. INTEGER AND SET PARTITIONS

We define an integer partition  $\lambda$  to be a finite multiset of positive integers. Thus the multiset  $\lambda = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$  is a partition of  $n$  if  $\lambda_1 + \lambda_2 + \dots + \lambda_k = n$ . Sometimes it will be necessary to consider the multiplicity of the elements of the partition  $\lambda$ . We then write

$$\lambda = \{\lambda_1^{m_1}, \lambda_2^{m_2}, \dots, \lambda_p^{m_p}\},$$

where we tacitly assume that  $\lambda_i \neq \lambda_j$  for two different indices  $i \neq j$ .

Let  $I_n$  be the set of all integer partitions of  $n$ . We form a poset on these integer partitions where the cover relation is given by adding two parts. In terms of multisets the cover relation is

$$\{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k\} \prec \{\lambda_1 + \lambda_2, \lambda_3, \dots, \lambda_k\}.$$

Note that the partition  $\{1, 1, \dots, 1\}$  is the minimal element and  $\{n\}$  is the maximal element in the partial order.

Let  $\Pi_n$  denote the poset of all set partitions of  $[n] = \{1, 2, \dots, n\}$  where the partial order is given by merging blocks, that is,

$$\{B_1, B_2, B_3, \dots, B_k\} \prec \{B_1 \cup B_2, B_3, \dots, B_k\}.$$

The poset  $\Pi_n$  is in fact a lattice, called the partition lattice. Let  $|\pi|$  denote the number of blocks of the partition  $\pi$ . Furthermore, for a set partition  $\pi = \{B_1, B_2, \dots, B_k\}$  define its type to be the integer partition of  $n$  given by the multiset  $\text{type}(\pi) = \{|B_1|, |B_2|, \dots, |B_k|\}$ .

The symmetric group  $\mathfrak{S}_n$  acts on subsets of  $[n]$  by relabeling the elements. Similarly, the symmetric group  $\mathfrak{S}_n$  acts on the partition lattice by relabeling the elements of the blocks. For  $\pi = \{B_1, B_2, \dots, B_k\}$  a set partition the action is given by  $\alpha \cdot \pi = \{\alpha(B_1), \alpha(B_2), \dots, \alpha(B_k)\}$ . Finally, when we speak about the action of the symmetric group  $\mathfrak{S}_{n-1}$ , we view the group  $\mathfrak{S}_{n-1}$  as the subgroup  $\{\alpha \in \mathfrak{S}_n : \alpha_n = n\}$  of the symmetric group  $\mathfrak{S}_n$ .

### 3. COMPOSITIONS AND ORDERED SET PARTITIONS

A composition  $\vec{c} = (c_1, c_2, \dots, c_k)$  of  $n$  is an ordered list of positive integers such that  $c_1 + c_2 + \dots + c_k = n$ . Let  $\text{Comp}(n)$  be the set of all compositions of  $n$ . We make  $\text{Comp}(n)$  into a poset by introducing the cover relation given by adding adjacent entries, that is,

$$(c_1, \dots, c_i, c_{i+1}, \dots, c_k) \prec (c_1, \dots, c_i + c_{i+1}, \dots, c_k).$$

The poset  $\text{Comp}(n)$  is isomorphic to the Boolean algebra on  $n - 1$  elements. Note that  $(1, 1, \dots, 1)$  and  $(n)$  are the minimal and maximal elements of  $\text{Comp}(n)$ , respectively. Define the type of a composition  $\vec{c} = (c_1, c_2, \dots, c_k)$  to be the integer partition  $\text{type}(\vec{c}) = \{c_1, c_2, \dots, c_k\}$  of  $n$ . Furthermore, let  $|\vec{c}|$  denote the number of parts of the composition  $\vec{c}$ .

For a composition  $\vec{c} = (c_1, c_2, \dots, c_k)$  of  $n$ , the *multinomial coefficient* is given by

$$\binom{n}{\vec{c}} = \binom{n}{c_1, c_2, \dots, c_k} = \frac{n!}{c_1! \cdot c_2! \cdot \dots \cdot c_k!}.$$

For  $\alpha \in \mathfrak{S}_n$ , let the descent set of  $\alpha$ , denoted by  $\text{Des}(\alpha)$ , be the subset of  $[n - 1]$  given by  $\text{Des}(\alpha) = \{i \in [n - 1] : \alpha(i) > \alpha(i + 1)\}$ . Throughout this paper it will be more convenient to consider  $\text{Des}(\alpha)$  as a composition of  $n$ , namely, if  $\text{Des}(\alpha) = \{i_1 < i_2 < \dots < i_k\}$ , then we consider  $\text{Des}(\alpha)$  as a composition of  $n$  given by  $\text{Des}(\alpha) = (i_1, i_2 - i_1, \dots, i_k - i_{k-1}, n - i_k)$ . Note that the identity permutation  $(1, 2, \dots, n)$  has descent composition  $(n)$ .

Let  $\beta_n(\vec{c})$  be the number of permutations  $\alpha$  in  $\mathfrak{S}_n$  such that  $\text{Des}(\alpha) = \vec{c}$ . Likewise, define  $\beta_n^*(\vec{c})$  to be the number of permutations  $\alpha$  in  $\mathfrak{S}_n$  with descent composition  $\vec{c}$  and  $\alpha(n) = n$ . Observe that

$$(3.1) \quad \binom{n-1}{c_1, \dots, c_{k-1}, c_k - 1} = \sum_{\substack{\vec{d} \in \text{Comp}(n) \\ \vec{c} \leq \vec{d}}} \beta_n^*(\vec{d}).$$

An *ordered set partition*  $\sigma = (C_1, C_2, \dots, C_p)$  of  $[n]$  is a list of non-empty blocks such that the set  $\{C_1, C_2, \dots, C_p\}$  is a partition of the set  $[n]$ , where the order of the blocks now matters. Let  $|\sigma|$  denote the number of blocks in the ordered set partition  $\sigma$ .

Let  $Q_n$  be the set of all ordered set partitions on the set  $[n]$ . Introduce a partial order on  $Q_n$  where the cover relation is joining adjacent blocks, that is,

$$(C_1, \dots, C_i, C_{i+1}, \dots, C_p) \prec (C_1, \dots, C_i \cup C_{i+1}, \dots, C_p).$$

Observe that the poset  $Q_n$  has the maximal element  $([n])$ , along with  $n!$  minimal elements, namely the ordered set partitions  $(\{\alpha_1\}, \{\alpha_2\}, \dots, \{\alpha_n\})$ , one for each permutation  $\alpha_1\alpha_2\cdots\alpha_n \in \mathfrak{S}_n$ . Moreover, every interval in  $Q_n$  is isomorphic to a Boolean algebra.

Define the *type* of an ordered set partition  $\sigma = (C_1, C_2, \dots, C_k)$  to be the composition of  $n$  given by  $\text{type}(\sigma) = (|C_1|, |C_2|, \dots, |C_k|)$ .

**Definition 3.1.** For a permutation  $\alpha \in \mathfrak{S}_n$  and a composition  $\vec{d} = (d_1, d_2, \dots, d_k)$  of  $n$ , let  $\sigma(\alpha, \vec{d})$  denote the unique ordered set partition in  $Q_n$  of type  $\vec{d}$  whose elements are given, in order, by the permutation  $\alpha$ , that is,

$$\sigma(\alpha, \vec{d}) = (\{\alpha(1), \dots, \alpha(d_1)\}, \{\alpha(d_1 + 1), \dots, \alpha(d_2)\}, \dots, \{\alpha(d_{k-1} + 1), \dots, \alpha(n)\}).$$

Finally, the symmetric group  $\mathfrak{S}_n$  acts on ordered set partitions by relabeling, that is

$$\alpha \cdot (C_1, C_2, \dots, C_k) = (\alpha(C_1), \alpha(C_2), \dots, \alpha(C_k)).$$

#### 4. TOPOLOGICAL CONSIDERATIONS

Let  $P$  be a poset. Recall the order complex of  $P$ , denoted  $\Delta(P)$ , is the simplicial complex whose  $i$ -dimensional faces are the chains in  $P$  with  $i + 1$  elements. If  $P$  has a minimal element  $\hat{0}$  or a maximal element  $\hat{1}$ , then  $\Delta(P)$  is a contractible complex. Thus we will be removing these elements to ensure interesting topology.

Recall a simplicial complex  $\Delta$  is a finite collection of sets such that the empty set belongs to  $\Delta$  and  $\Delta$  is closed under inclusion. We will find it easier to view a simplicial complex as a partially ordered set  $\Delta$  such that (i)  $\Delta$  has a unique minimal element  $\hat{0}$  and (ii) every interval  $[\hat{0}, x]$  for  $x \in \Delta$  is isomorphic to a Boolean algebra. A poset  $P$  satisfying these conditions is called a *simplicial poset*. Notice that a poset  $P$  is simplicial if  $P$  is the face poset of a simplicial complex. Furthermore, note that the second condition in the definition of a simplicial poset makes the poset  $\Delta$  ranked since every saturated chain between the minimal element  $\hat{0}$  and an element  $x$  has the same length. Thus the *dimension* of an element  $x$  is defined by its rank minus one, that is,  $\dim(x) = \rho(x) - 1$ .

A filter in a poset  $P$  is an upper order ideal. Hence if  $F$  is a filter in  $P$ , then the dual filter  $F^*$  in the dual poset  $P^*$  is now a lower order ideal. In particular, if  $\Delta \subseteq \text{Comp}(n)$  is a filter, since upper order ideals in  $\text{Comp}(n)$  are isomorphic to Boolean algebras, the dual of  $\Delta$  is a simplicial poset in the dual space  $\text{Comp}(n)^*$ , which has cover relation given by splitting rather than merging. To emphasize that we have dualized, we use  $\leq^*$  to denote the order relation in the dualized  $\text{Comp}(n)$ .

Lastly, the *link* of a face  $F$  in a simplicial complex  $\Delta$  is given by  $\text{lk}_F(\Delta) = \{G \in \Delta : F \cup G \in \Delta, F \cap G = \emptyset\}$ . However, working with the poset definition of a simplicial complex, we have the following equivalent definition of the link. The link is the principle filter generated by the face  $x$ , that is,  $\text{lk}_x(\Delta) = \{y \in \Delta : x \leq y\}$ . One advantage of this definition is that we do not have to relabel the faces when considering the link.

From now on our simplicial complex  $\Delta$  will be a filter in the composition lattice  $\text{Comp}(n)$ , with the dual order  $\leq^*$ .

Let  $C_k(\text{Comp}(n))$  be the linear span over  $\mathbb{C}$  of all compositions of  $n$  into  $k + 2$  parts. We obtain a chain complex by defining the boundary map as follows. Define the map  $\partial_{k,j} : C_k(\text{Comp}(n)) \rightarrow C_{k-1}(\text{Comp}(n))$  by

$$\partial_{k,j}(c_1, \dots, c_j, c_{j+1}, \dots, c_{k+2}) = (c_1, \dots, c_j + c_{j+1}, \dots, c_{k+2}).$$

Then the boundary map on  $\text{Comp}(n)$  is given by  $\partial_k = \sum_{j=1}^{k+1} (-1)^{j-1} \cdot \partial_{k,j}$

Consider the dual order on the set of ordered set partitions  $Q_n$ . For  $\Delta \subseteq \text{Comp}(n)$  a complex, let  $Q_\Delta = \{\tau \in Q_n : \text{type}(\tau) \in \Delta\}$ . The filter  $Q_\Delta$  is also a simplicial poset, so we refer to  $Q_\Delta$  as a complex.

Define  $C_k(Q_n)$  to be the linear span over  $\mathbb{C}$  of all ordered set partitions of  $[n]$  with  $k+2$  blocks. The boundary map  $\partial_k : C_k(Q_n) \rightarrow C_{k-1}(Q_n)$  on  $Q_n$  is given by  $\partial_k(\sigma(\alpha, \vec{d})) = \sigma(\alpha, \partial_k(\vec{d}))$ , where  $\partial_k(\vec{d})$  is the boundary map applied to the composition  $\vec{d}$  in  $C_k(\text{Comp}(n))$ , and where  $\sigma(\alpha, \vec{c})$  is given in Definition 3.1. This boundary map is inherited by the subcomplex  $Q_\Delta$ .

Finally, for simplicial complexes  $\Delta$  and  $\Gamma$  in  $\text{Comp}(n)$  and  $\text{Comp}(m)$  respectively, their *join* is defined to be poset

$$\Delta * \Gamma = \{\vec{c} \circ \vec{d} : \vec{c} \in \Delta, \vec{d} \in \Gamma\},$$

where  $\circ$  denote the concatenation of compositions. Note that the join  $\Delta * \Gamma$  has the composition  $(n, m)$  as its minimal element. Furthermore, we have the following basic lemma on Morse matchings of joins of complexes.

**Lemma 4.1.** *Let  $\Delta$  and  $\Gamma$  be two complexes in  $\text{Comp}(m)$  and  $\text{Comp}(m)$  respectively, each having a discrete Morse matching. Let  $\Delta^c$  and  $\Gamma^c$  be the sets of critical cells of  $\Delta$  and  $\Gamma$ , respectively. Then the join  $\Delta * \Gamma$  has a Morse matching where the critical cells are*

$$\{\vec{c} \circ \vec{d} : \vec{c} \in \Delta^c, \vec{d} \in \Gamma^c\}.$$

*Proof.* Define a matching of the join  $\Delta * \Gamma$  as follows. If  $\vec{c} \prec \vec{c}'$  is an edge in the discrete Morse matching of  $\Delta$  and  $\vec{d} \in \Gamma$  then match  $\vec{c} \circ \vec{d} \prec \vec{c}' \circ \vec{d}$ . If  $\vec{c}$  is a critical cell of  $\Delta$  and  $\vec{d} \prec \vec{d}'$  is an edge in the discrete Morse matching of  $\Gamma$  then match  $\vec{c} \circ \vec{d} \prec \vec{c} \circ \vec{d}'$ . It is straightforward to verify that this matching is acyclic and that the set of critical cells is as described.  $\square$

## 5. BORDER STRIPS AND SPECHT MODULES

A border strip  $B$  is a connected skew-shape which does not contain a two by two square. For each composition  $\vec{c} = (c_1, c_2, \dots, c_k)$  there is a unique border strip such that the number of boxes in the  $i$ th row is given by  $c_i$  and every two adjacent rows overlap in one position. Denote this border strip by  $B(\vec{c})$ . See Figure 1 for an example.

In an analogous fashion, for a composition  $\vec{c} \in \text{Comp}(n)$  we define the border shape  $A(\vec{c})$  to be the skew-shape whose  $i$ th row has length  $c_i$  such that the rows of  $A(\vec{c})$  are non-overlapping.

Let  $R_i$  be the interval  $[c_1 + \dots + c_{i-1} + 1, c_1 + \dots + c_{i-1} + c_i]$ . The row stabilizer of the border strip  $B(\vec{c})$  is the subgroup  $\mathfrak{S}_{R_1} \times \mathfrak{S}_{R_2} \times \dots \times \mathfrak{S}_{R_k}$  of the symmetric group  $\mathfrak{S}_n$ .

Since the poset  $\text{Comp}(n)$  of all compositions of  $n$  is a isomorphic to Boolean algebra, every composition has a complementary composition  $\vec{c}^c$ . To obtain the complement of composition write every part of the composition as a sum of 1s where we separate the parts with commas. Then the complement is obtained by exchanging the plus signs and the commas. Similarly, the column stabilizer is defined as the row stabilizer of the border strip of the complementary composition. More precisely, let  $(d_1, d_2, \dots, d_p)$  be the complementary composition  $\vec{c}^c$  and let  $K_i$  be the interval  $[d_1 + \dots + d_{i-1} + 1, d_1 + \dots + d_{i-1} + d_i]$ . Then the column stabilizer is the subgroup  $\mathfrak{S}_{K_1}^C = \mathfrak{S}_{K_1} \times \mathfrak{S}_{K_2} \times \dots \times \mathfrak{S}_{K_p}$ . See Figure 1.

We now review some basic representation theory of the symmetric group. For a less terse introduction, see [13, Chapter 3]. A border strip tableau  $t$  of shape  $\vec{c}$  is a filling of the border strip  $B(\vec{c})$ . We say a tableau  $t$  is *standard* if the entries of  $t$  are increasing along the rows from left to right and increasing down the columns. A border strip tabloid, denoted  $[t]$ , is a border strip

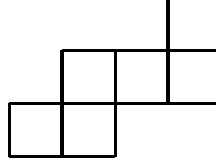


FIGURE 1. The border strip  $B(\vec{c})$  associated with the composition  $\vec{c} = (2, 3, 1)$ . The row stabilizer is the group  $\mathfrak{S}_{\vec{c}} = \mathfrak{S}_{[1,2]} \times \mathfrak{S}_{[3,5]} \times \mathfrak{S}_{[6,6]}$ . Note that  $\vec{c} = (1+1, 1+1+1, 1)$  so that the complementary composition is  $\vec{c}^c = (1, 1+1, 1, 1+1) = (1, 2, 1, 2)$ . Hence the column stabilizer is  $\mathfrak{S}_{\vec{c}}^C = \mathfrak{S}_{[1,1]} \times \mathfrak{S}_{[2,3]} \times \mathfrak{S}_{[4,4]} \times \mathfrak{S}_{[5,6]}$ .

tableau under row equivalence. Define the permutation module,  $M^{B(\vec{c})}$ , to be the vector space with basis elements given by all tabloids of shape  $B(\vec{c})$ . A polytabloid is defined by the alternating sum  $e_t = \sum_{\gamma \in \mathfrak{S}_{\vec{c}}^C} (-1)^\gamma \cdot [\gamma \cdot t]$ , where  $\mathfrak{S}_{\vec{c}}^C$  is the column stabilizer of the tableau  $t$  of shape  $\vec{c}$ . Lastly, the Specht module, denoted  $S^{B(\vec{c})}$ , is the subspace of  $M^{B(\vec{c})}$  generated by polytabloids. The dimension of the Specht module  $S^{B(\vec{c})}$  is given by the descent set statistics  $\beta_n(\vec{c})$ , while the dimension of the permutation module  $M^{B(\vec{c})}$  is given by the multinomial coefficient  $\binom{n}{\vec{c}}$ .

We now define two operations on compositions. The motivation comes from the associated Specht and permutation modules. For a composition  $\vec{c} = (c_1, \dots, c_{k-1}, c_k)$  let  $\vec{c}-1$  denote the composition  $(c_1, \dots, c_{k-1}, c_k - 1)$  if  $c_k \geq 2$ , and otherwise let  $\vec{c}-1$  denote the empty composition. Similarly, let  $\vec{c}/1$  denote the composition  $(c_1, \dots, c_{k-1}, c_k - 1)$  if  $c_k \geq 2$ , and otherwise let  $\vec{c}/1$  denote the composition  $(c_1, \dots, c_{k-1})$ . Note that if  $\vec{c}$  is a composition of  $n$  then  $\vec{c}/1$  is always a composition of  $n-1$ .

For a composition  $\vec{c}$  of  $n$  let  $B^*(\vec{c})$  denote the border strip  $B(\vec{c}-1)$ . All our results of this paper are stated in terms of the Specht modules  $S^{B^*(\vec{c})}$  where the group action is by  $\mathfrak{S}_{n-1}$ . We think of this Specht module as a submodule of  $S^{B(\vec{c})}$  spanned by all standard Young tableaux where the northeastern-most box is filled with  $n$ . Note that when the composition ends with the entry 1, there are no such standard Young tableaux, and hence  $S^{B^*(\vec{c})}$  is the zero module.

For a composition  $\vec{c}$  define the two shapes  $B^\#(\vec{c}) = B(\vec{c}/1)$  and  $A^\#(\vec{c}) = A(\vec{c}/1)$ . Observe that the permutation module  $M^{B^\#(\vec{c})}$  is a submodule of  $M^{B(\vec{c})}$ . That is, the span of the tabloids in  $M^{B(\vec{c})}$  where the tabloids has the filling  $n$  in the northeastern-most box is the module  $M^{B^\#(\vec{c})}$ .

Furthermore, the dimensions of the Specht module  $S^{B^*(\vec{c})}$  and the permutation module  $M^{B^\#(\vec{c})}$  are  $\beta_n^*(\vec{c})$  and  $\binom{n}{\vec{c}/1}$ , respectively. Additionally, we have the decomposition

$$M^{B^\#(\vec{c})} \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \leq \vec{d}} S^{B^*(\vec{d})},$$

which is the representation theoretic analogue of equation (3.1). See Lemma 13.3 for a proof.

## 6. THE ORDERED PARTITION FILTER $Q_\Delta^*$

We now introduce the ordered partition filter  $Q_\Delta^*$ . This filter will serve us as an important stepping stone to understanding the topology of general filters in the partition lattice. The transition from  $Q_\Delta^*$  to the partition lattice uses Quillen's Fiber Lemma; see Section 11. Note that by considering the reverse orders in  $\text{Comp}(n)$  and in  $Q_n$  we obtain two simplicial posets. Hence for  $\Delta$  a non-empty filter in  $\text{Comp}(n)$ , we view  $\Delta$  as a simplicial complex under the reverse order  $\leq^*$ . See the discussion in Section 4.

**Definition 6.1.** Let  $\Delta$  be a filter in  $\text{Comp}(n)$ , that is,  $\Delta$  is a simplicial complex consisting of compositions of  $n$ . Define the ordered partition filter  $Q_\Delta^*$  to be all ordered set partitions whose type is in the complex  $\Delta$  and whose last block contains the element  $n$ , that is,

$$Q_\Delta^* = \{\sigma = (C_1, C_2, \dots, C_k) \in Q_n : \text{type}(\sigma) \in \Delta, n \in C_k\}.$$

Note that we view  $Q_\Delta^*$  as a simplicial complex. Our purpose is to study the reduced homology groups of this complex.

Recall that the link of a composition  $\vec{c}$  in  $\Delta$  is the filter

$$\text{lk}_{\vec{c}}(\Delta) = \{\vec{d} \in \Delta : \vec{d} \leq^* \vec{c}\},$$

where  $\leq^*$  is the reverse of the partial order of  $\text{Comp}(n)$ . Since  $\text{lk}_{\vec{c}}(\Delta)$  is now a simplicial poset with minimal element  $\vec{c}$ , we have a dimension shift from  $\Delta$  to  $\text{lk}_{\vec{c}}(\Delta)$  given by

$$(6.1) \quad \dim_{\text{lk}_{\vec{c}}(\Delta)}(\vec{d}) = \dim_\Delta(\vec{d}) - |\vec{c}| + 1$$

for  $\vec{d} \in \text{lk}_{\vec{c}}(\Delta)$ .

**Remark 6.2.** The symmetric group  $\mathfrak{S}_{n-1}$  acts on  $Q_\Delta^*$  by permutation, whereas the action of  $\mathfrak{S}_{n-1}$  on the complex  $\Delta$  is the trivial action. Furthermore, the type map from  $Q_\Delta^*$  to  $\Delta$  respects this action, since the two ordered set partitions  $\sigma$  and  $\tau \cdot \sigma$  have the same type.

A special case of  $Q_\Delta^*$  is when the simplicial complex  $\Delta$  is a simplex, that is,  $\Delta$  is generated by one composition  $\vec{c}$ . This case was studied by Ehrenborg and Jung in [7]. Their results are given below.

**Theorem 6.3** (Ehrenborg–Jung). *Let  $\vec{c}$  be a composition of  $n$  into  $k$  parts. Then the complex  $Q_{\vec{c}}^*$  is a wedge of  $\beta_n^*(\vec{c})$  spheres of dimension  $k - 2$ . Furthermore, the top homology group  $\tilde{H}_{k-2}(Q_{\vec{c}}^*)$  is isomorphic to the Specht module  $S^{B^*(\vec{c})}$  as an  $\mathfrak{S}_{n-1}$ -module. This isomorphism  $\phi : S^{B^*(\vec{c})} \rightarrow \tilde{H}_{k-2}(Q_{\vec{c}}^*)$  is given by*

$$\phi(e_t) = \sum_{\gamma \in \mathfrak{S}_{\vec{c}}^C} (-1)^\gamma \cdot \sigma(\alpha \cdot \gamma, \vec{c}),$$

where the permutation  $\alpha \in \mathfrak{S}_n$  is obtained by reading the entries of the tabloid  $t$  from southwest to northeast and attaching the element  $n$  at the end.

Note that Ehrenborg and Jung formulated their result in terms of pointed set partitions. That is, our notation  $Q_{\vec{c}}^*$  is  $\Delta_{\vec{d}}$  in their notation, where  $\vec{d} = (c_1, \dots, c_{k-1}, c_k - 1)$ . They allow the last entry of a composition to be zero and similarly the last entry of an ordered set partition to be empty. Moreover, our notation  $\Pi_{\vec{c}}^*$  is in their notation  $\Pi_{\vec{d}}^\bullet$ .

We can now state the main result of this section.

**Theorem 6.4.** *Let  $\Delta$  be a simplicial complex of compositions of  $n$ . Then the  $i$ th reduced homology group of the simplicial complex  $Q_\Delta^*$  is given by*

$$\tilde{H}_i(Q_\Delta^*) \cong \bigoplus_{\vec{c} \in \Delta} \tilde{H}_{i-|\vec{c}|+1}(\text{lk}_{\vec{c}}(\Delta)) \otimes S^{B^*(\vec{c})}.$$

Furthermore, this isomorphism holds as  $\mathfrak{S}_{n-1}$ -modules.

We will prove Theorem 6.4 in Sections 7 through 9.

7. THE HOMOMORPHISM  $\phi_i^\Delta$ 

In this section and the next two sections we present a proof of Theorem 6.4. The major step is to show that if Theorem 6.4 holds for  $\Delta$ ,  $\Gamma$ , and the intersection  $\Delta \cap \Gamma$ , then it also holds for the union  $\Delta \cup \Gamma$ . This step requires Mayer–Vietoris sequences. When  $\Delta$  is generated by a single composition  $\vec{c}$  in  $\text{Comp}(n)$ , the result follows from Theorem 6.3. Finally, since any simplicial complex is a union of simplices, Theorem 6.4 will hold for arbitrary simplicial complexes  $\Delta$  in  $\text{Comp}(n)$ .

We begin by defining the isomorphism of Theorem 6.4 explicitly. Throughout the paper we will let  $i_{\vec{c}}$  denote the shift  $i - |\vec{c}| + 1$ .

**Definition 7.1.** Let  $D_i^{\vec{c}}(\Delta)$  be the tensor product  $C_{i_{\vec{c}}}(\text{lk}_{\vec{c}}(\Delta)) \otimes M^{B^\#(\vec{c})}$  where  $C_j(\text{lk}_{\vec{c}}(\Delta))$  is the  $j$ th chain group of the link  $\text{lk}_{\vec{c}}(\Delta)$ . Let  $D^{\vec{c}}(\Delta)$  be the chain complex whose  $i$ th chain group is  $D_i^{\vec{c}}(\Delta)$  and whose boundary map is  $\partial \otimes \text{id}$ . Lastly, let  $D(\Delta)$  be the chain complex with  $i$ th chain group  $\bigoplus_{\vec{c} \in \Delta} D_i^{\vec{c}}(\Delta)$  with the differential  $\bigoplus_{\vec{c} \in \Delta} \partial \otimes \text{id}$ .

**Definition 7.2.** Define the chain complex  $E^{\vec{c}}(\Delta)$  analogous to  $D^{\vec{c}}(\Delta)$  of Definition 7.1 above by replacing the permutation module  $M^{B^\#(\vec{c})}$  with the Specht module  $S^{B^*(\vec{c})}$ . We also have the corresponding chain complex  $E(\Delta)$  with the same differential.

**Lemma 7.3.** The homology of the chain complexes  $D(\Delta)$  and  $E(\Delta)$  are given by

$$\begin{aligned} \tilde{H}_i(D(\Delta)) &\cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in \Delta} \tilde{H}_{i_{\vec{c}}}(\text{lk}_{\vec{c}}(\Delta)) \otimes M^{B^\#(\vec{c})}, \\ \tilde{H}_i(E(\Delta)) &\cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in \Delta} \tilde{H}_{i_{\vec{c}}}(\text{lk}_{\vec{c}}(\Delta)) \otimes S^{B^*(\vec{c})}. \end{aligned}$$

*Proof.* The homology of the chain complex  $D^{\vec{c}}(\Delta)$  is given by  $\ker(\partial_{i_{\vec{c}}} \otimes \text{id}) / \text{im}(\partial_{i_{\vec{c}+1}} \otimes \text{id}) \cong (\ker(\partial_{i_{\vec{c}}}) \otimes M^{B^\#(\vec{c})}) / (\text{im}(\partial_{i_{\vec{c}+1}}) \otimes M^{B^\#(\vec{c})}) \cong \ker(\partial_{i_{\vec{c}}}) / \text{im}(\partial_{i_{\vec{c}+1}}) \otimes M^{B^\#(\vec{c})} \cong \tilde{H}_{i_{\vec{c}}}(\text{lk}_{\vec{c}}(\Delta)) \otimes M^{B^\#(\vec{c})}$ . The analogous result holds for  $E^{\vec{c}}(\Delta)$  and the lemma follows by taking direct sums.  $\square$

For the rest of this section we let  $t$  denote a tabloid in the permutation module  $M^{B^\#(\vec{c})}$  and  $\alpha \in \mathfrak{S}_n$  is the permutation obtained by reading the entries of the tabloid  $t$  in increasing order from southwest to northeast and adjoining the element  $n$  at the end.

**Definition 7.4.** The  $\mathfrak{S}_{n-1}$ -action on  $D_i^{\vec{c}}(\Delta)$  is given by  $\tau \cdot (\vec{d} \otimes t) = \vec{d} \otimes (\tau \circ t)$ , for  $\tau \in \mathfrak{S}_{n-1}$  and  $\vec{d} \otimes t$  a basis element of  $D_i^{\vec{c}}(\Delta) = C_{i_{\vec{c}}}(\text{lk}_{\vec{c}}(\Delta)) \otimes M^{B^\#(\vec{c})}$ .

Notice that Definition 7.4 states that  $\mathfrak{S}_{n-1}$  acts on  $D_i^{\vec{c}}(\Delta)$  by acting trivially on the chain group  $C_{i_{\vec{c}}}(\text{lk}_{\vec{c}}(\Delta))$  and by relabeling on  $M^{B^\#(\vec{c})}$ .

**Definition 7.5.** For a simplicial complex  $\Delta$  and a composition  $\vec{c}$  in  $\Delta$  define the map

$$\phi_i^{\Delta, \vec{c}} : C_{i_{\vec{c}}}(\text{lk}_{\vec{c}}(\Delta)) \otimes M^{B^\#(\vec{c})} \longrightarrow C_i(Q_\Delta^*),$$

on basis elements by  $\phi_i^{\Delta, \vec{c}}(\vec{d} \otimes t) = \sigma(\alpha, \vec{d})$ .

Since  $\vec{d} \in C_{i_{\vec{c}}}(\text{lk}_{\vec{c}}(\Delta))$  is a basis element, we know that  $\vec{d}$  is a simplex of  $\text{lk}_{\vec{c}}(\Delta)$  of dimension  $i_{\vec{c}} = i - |\vec{c}| + 1$ , and thus by the dimension shift in equation (6.1), we have that  $|\vec{d}| = i + 2$ , so that  $\phi_i^{\Delta, \vec{c}}(\vec{d}) = \sigma(\pi, \vec{d})$  is an ordered partition of dimension  $i$ . Lastly, since tabloids in  $M^{B^\#(\vec{c})}$  have  $n$  in the last block, we are guaranteed that  $\phi_i^{\Delta, \vec{c}}(\vec{d}) \in C_i(Q_\Delta^*)$ .

**Lemma 7.6.** The map  $\phi_i^{\Delta, \vec{c}} : C_{i_{\vec{c}}}(\text{lk}_{\vec{c}}(\Delta)) \otimes M^{B^\#(\vec{c})} \longrightarrow C_i(Q_\Delta^*)$  respects the  $\mathfrak{S}_{n-1}$ -action.



*Proof.* Let  $\tau \in \mathfrak{S}_{n-1}$  and  $\vec{d} \otimes t$  be a basis element of  $C_{i_{\vec{c}}}(\text{lk}_{\vec{c}}(\Delta)) \otimes M^{B^\#(\vec{c})}$ . Then we have

$$\phi_i^{\Delta, \vec{c}}(\tau \cdot (\vec{d} \otimes t)) = \phi_i^{\Delta, \vec{c}}(\vec{d} \otimes (\tau \cdot t)) = \sigma(\tau \cdot \alpha, \vec{d}) = \tau \cdot \sigma(\alpha, \vec{d}) = \tau \cdot \phi_i^{\Delta, \vec{c}}(\vec{d} \otimes t). \quad \square$$

**Lemma 7.7.** *The map  $\phi_i^{\Delta, \vec{c}}$  is an equivariant chain map between the complexes  $D^{\vec{c}}(\Delta)$  and  $C_i(Q_\Delta^*)$ . That is, the following diagram commutes:*

$$\begin{array}{ccc} D_i^{\vec{c}}(\Delta) & \xrightarrow{\partial \otimes \text{id}} & D_{i-1}^{\vec{c}}(\Delta) \\ \downarrow \phi_i^{\Delta, \vec{c}} & & \downarrow \phi_{i-1}^{\Delta, \vec{c}} \\ C_i(Q_\Delta^*) & \xrightarrow{\partial} & C_{i-1}(Q_\Delta^*) \end{array}$$

*Proof.* The boundary map  $\partial$  of  $\text{Comp}(n)$  as well as the boundary map  $\partial$  of  $Q_\Delta^*$  are given in Section 4. Let  $\vec{d} \otimes t \in C_{i_{\vec{c}}}(\text{lk}_{\vec{c}}(\Delta)) \otimes M^{B^\#(\vec{c})}$ . Tracing first right then down we obtain:

$$\phi_{i-1}^{\Delta, \vec{c}} \circ (\partial \otimes \text{id})(\vec{d} \otimes t) = \phi_{i-1}^{\Delta, \vec{c}}(\partial(\vec{d}) \otimes t) = \sigma(\alpha, \partial(\vec{d})).$$

Next, we trace down then right to obtain the same result:

$$\partial \circ \phi_i^{\Delta, \vec{c}}(\vec{d} \otimes t) = \partial(\sigma(\alpha, \vec{d})) = \sigma(\alpha, \partial(\vec{d})).$$

The equivariance of  $\phi_i^{\Delta, \vec{c}}$  is a consequence of Lemma 7.6.  $\square$

**Lemma 7.8.** *The map  $\phi_i^{\Delta, \vec{c}}$  induces a map*

$$\phi_i^{\Delta, \vec{c}} : \widetilde{H}_{i_{\vec{c}}}(\text{lk}_{\vec{c}}(\Delta)) \otimes M^{B^\#(\vec{c})} \longrightarrow \widetilde{H}_i(Q_\Delta^*)$$

*given by  $\phi_i^{\Delta}(\vec{d} \otimes t) = \overline{\sigma(\alpha, \vec{d})}$ , for  $\vec{d} \in C_{i_{\vec{c}}}(\text{lk}_{\vec{c}}(\Delta))$  a cycle.*

*Proof.* Since  $\phi_i^{\Delta, \vec{c}}$  is an equivariant chain map between the chain complexes  $D^{\vec{c}}(\Delta)$  and  $C_i(Q_\Delta^*)$  by Lemma 7.7, the result follows.  $\square$

For the rest of the paper, the use of the bar to indicate the quotient in passing from the chain space to the homology group will be suppressed for ease of notation.

**Definition 7.9.** *Define the map  $\phi_i^\Delta$  from  $D_i(\Delta) = \bigoplus_{\vec{c} \in \Delta} D_{i_{\vec{c}}}^{\vec{c}}(\Delta)$  to  $C_i(Q_\Delta^*)$  by adding all the  $\phi_i^{\Delta, \vec{c}}$  maps together, that is,*

$$(7.1) \quad \phi_i^\Delta = \sum_{\vec{c} \in \Delta} \phi_i^{\Delta, \vec{c}}.$$

Observe that  $\phi_i^\Delta$  restricts to a map from  $E_i(\Delta)$  to  $C_i(Q_\Delta^*)$ . Therefore  $\phi_i^\Delta$  also induces a map from  $\widetilde{H}_i(E(\Delta)) = \bigoplus_{\vec{c} \in \Delta} \widetilde{H}_{i_{\vec{c}}}(\text{lk}_{\vec{c}}(\Delta)) \otimes S^{B^*(\vec{c})}$  to  $\widetilde{H}_i(Q_\Delta^*)$  using Lemma 7.8.

## 8. THE MAIN THEOREM

We can now explicitly state the isomorphism of Theorem 6.4. First we introduce notation for the right-hand side of this theorem.

**Definition 8.1.** *Let  $K_i(\Delta)$  denote the direct sum  $\bigoplus_{\vec{c} \in \Delta} \widetilde{H}_{i_{\vec{c}}}(\text{lk}_{\vec{c}}(\Delta)) \otimes S^{B^*(\vec{c})}$ .*

A sharpening of Theorem 6.4 is the following result.

**Theorem 8.2.** *Let  $\Delta$  be a subcomplex of  $\text{Comp}(n)$ . Then the map*

$$\phi_i^\Delta : K_i(\Delta) \longrightarrow \widetilde{H}_i(Q_\Delta^*)$$

*is an  $\mathfrak{S}_{n-1}$ -equivariant isomorphism.*

Note that Lemma 7.3 tells us that the homology of the complex  $E(\Delta)$  is  $K(\Delta)$ , that is, for all  $i$  we have  $\tilde{H}_i(E(\Delta)) \cong_{\mathfrak{S}_{n-1}} K_i(\Delta)$ . Lemma 7.7 implies that equation (7.1) is a well-defined map from the homology of  $E(\Delta)$  to the homology groups  $\tilde{H}_i(Q_\Delta^*)$ .

We first prove Theorem 8.2 in the case when  $\Delta$  is a simplex. This is the case when  $\Delta$  is generated by one composition.

**Proposition 8.3.** *Assume that  $\Delta$  is a filter in  $\text{Comp}(n)$  generated by one composition, that is,  $\Delta$  is a simplex. Then Theorem 8.2 holds for  $\Delta$ .*

*Proof.* Suppose that  $\Delta \subseteq \text{Comp}(n)$  is generated by the composition  $\vec{d} = (d_1, d_2, \dots, d_k)$ . Theorem 6.3 states that  $Q_\Delta^*$  only has reduced homology in dimension  $k-2$ . Additionally, it states that the action of  $\mathfrak{S}_{n-1}$  on the top homology of  $Q_\Delta^*$  is given by the border shape Specht module  $S^{B^*(\vec{c})}$ , that is,  $\tilde{H}_{k-2}(Q_\Delta) \cong_{\mathfrak{S}_{n-1}} S^{B^*(\vec{c})}$ .

Next we show that  $\phi_i^\Delta : K_i(\Delta) \rightarrow \tilde{H}_i(Q_\Delta^*)$  is an isomorphism for all  $i$ . When  $i \neq k-2$  both sides are the trivial module, that is,  $K_i(\Delta) = 0 = \tilde{H}_i(Q_\Delta^*)$  and the map  $\phi_i^\Delta$  is directly an isomorphism. Now assume that  $i = k-2$ . Since all the links  $\text{lk}_{\vec{c}}(\Delta)$  for  $\vec{c} <^* \vec{d}$  are contractible, we have

$$K_{k-2}(\Delta) = \bigoplus_{\vec{c} \in \Delta} \tilde{H}_{k-2-|\vec{c}|+1}(\text{lk}_{\vec{c}}(\Delta)) \otimes S^{B^*(\vec{c})} = \tilde{H}_{-1}(\text{lk}_{\vec{d}}(\Delta)) \otimes S^{B^*(\vec{d})}.$$

Notice that  $\text{lk}_{\vec{d}}(\Delta)$  consists only of the composition  $\vec{d}$  itself, so that the  $(-1)$ -dimensional reduced homology group  $\tilde{H}_{-1}(\text{lk}_{\vec{d}}(\Delta))$  is the homology of the chain space  $C_{-1}(\text{lk}_{\vec{d}}(\Delta))$ , which is the one dimensional vector space with the generator  $\vec{d}$ . Therefore, the map  $\phi_{k-2}^\Delta : \tilde{H}_{-1}(\text{lk}_{\vec{d}}(\Delta)) \otimes S^{B^*(\vec{d})} \rightarrow \tilde{H}_{k-2}(Q_\Delta^*)$  is given by

$$\vec{d} \otimes e_t = \vec{d} \otimes \left( \sum_{\gamma \in \mathfrak{S}_{\vec{d}}^C} (-1)^\gamma \cdot [\gamma \cdot t] \right) \mapsto \sum_{\gamma \in \mathfrak{S}_{\vec{d}}^C} (-1)^\gamma \cdot \sigma(\alpha \cdot \gamma, \vec{d}).$$

But this is an isomorphism by Theorem 6.3.  $\square$

As a direct corollary we have that Theorem 8.2 holds for the empty simplex  $\{(n)\} \subseteq \text{Comp}(n)$ .

**Corollary 8.4.** *Theorem 8.2 holds for the empty simplicial complex, that is, the simplicial complex consisting only of the composition  $(n)$ .*

*Proof.* Apply Proposition 8.3 to the simplicial complex  $\Delta$  generated by the composition  $(n)$  in  $\text{Comp}(n)$ .  $\square$

## 9. THE BUILDING STEP

A simplicial complex which is not a simplex is the union of smaller simplicial complexes. We now prove that Theorem 8.2 holds for the complex  $\Delta \cup \Gamma$ , assuming that Theorem 8.2 holds for the simplicial complexes  $\Delta$ ,  $\Gamma$ , as well as the intersection  $\Delta \cap \Gamma$ . We build up in the isomorphism  $\phi_i^{\Delta \cup \Gamma}$  between  $K_i(\Delta \cup \Gamma)$  and  $\tilde{H}_i(Q_{\Delta \cup \Gamma}^*)$  from the associated isomorphisms holding for the smaller complexes.

**Lemma 9.1.** *The following two identities hold for the link:*

$$\text{lk}_{\vec{c}}(\Delta \cap \Gamma) = \text{lk}_{\vec{c}}(\Delta) \cap \text{lk}_{\vec{c}}(\Gamma) \quad \text{and} \quad \text{lk}_{\vec{c}}(\Delta \cup \Gamma) = \text{lk}_{\vec{c}}(\Delta) \cup \text{lk}_{\vec{c}}(\Gamma).$$

**Lemma 9.2.** *The following two identities hold for the ordered set partition poset:*

$$Q_{\Delta \cap \Gamma}^* = Q_{\Delta}^* \cap Q_{\Gamma}^* \quad \text{and} \quad Q_{\Delta \cup \Gamma}^* = Q_{\Delta}^* \cup Q_{\Gamma}^*.$$

The proofs of these two lemmas are straightforward and are omitted.

Before we begin the proof of Theorem 8.2, let us remind ourselves of Definition 7.1. For each composition  $\vec{c}$  in  $\Delta$  we have the chain complex  $D^{\vec{c}}(\Delta)$  whose  $i$ th chain group is  $D_i^{\vec{c}}(\Delta) = C_{i\vec{c}}(\text{lk}_{\vec{c}}(\Delta)) \otimes M^{B^{\#}(\vec{c})}$ . Furthermore,  $D(\Delta)$  is the chain complex obtained by taking the direct sum of  $D^{\vec{c}}(\Delta)$ , where  $\vec{c}$  ranges over all compositions in  $\Delta$ .

We now begin the proof of Theorem 8.2.

**Lemma 9.3.** *For  $\vec{c} \in \Delta \cap \Gamma$  the following diagram is commutative, and its rows are exact.*

$$\begin{array}{ccccccc} 0 & \longrightarrow & D_i^{\vec{c}}(\Delta \cap \Gamma) & \longrightarrow & D_i^{\vec{c}}(\Delta) \oplus D_i^{\vec{c}}(\Gamma) & \longrightarrow & D_i^{\vec{c}}(\Delta \cup \Gamma) \longrightarrow 0 \\ & & \downarrow \phi_i^{\Delta \cap \Gamma, \vec{c}} & & \downarrow \phi_i^{\Delta, \vec{c}} \oplus \phi_i^{\Gamma, \vec{c}} & & \downarrow \phi_i^{\Delta \cup \Gamma, \vec{c}} \\ 0 & \longrightarrow & C_i(Q_{\Delta \cap \Gamma}^*) & \longrightarrow & C_i(Q_{\Delta}^*) \oplus C_i(Q_{\Gamma}^*) & \longrightarrow & C_i(Q_{\Delta \cup \Gamma}^*) \longrightarrow 0 \end{array}$$

*Proof.* The horizontal maps in the above diagram are given by the construction of the Mayer–Vietoris sequence applied to  $\text{lk}_{\vec{c}}(\Delta \cup \Gamma) = \text{lk}_{\vec{c}}(\Delta) \cup \text{lk}_{\vec{c}}(\Gamma)$  in the top row, and  $Q_{\Delta \cup \Gamma}^* = Q_{\Delta}^* \cup Q_{\Gamma}^*$  in the bottom row. The top horizontal maps have also been tensored with the identity map on the Specht modules. As the Specht module is free, both the top and bottom rows of the diagram remain exact.

We show commutativity of the left square, as the right square is analogous. Let  $\vec{d} \otimes \alpha \in C_{i\vec{c}}(\Delta \cap \Gamma) \otimes M^{B^{\#}(\vec{c})}$  be a basis element. First we trace right then down to obtain:

$$\vec{d} \otimes \alpha \mapsto (\vec{d} \otimes \alpha) \oplus -(\vec{d} \otimes \alpha) \mapsto \sigma(\alpha, \vec{d}) \oplus -\sigma(\alpha, \vec{d}).$$

We obtain the same result by first tracing down then right:

$$\vec{d} \otimes \alpha \mapsto \sigma(\alpha, \vec{d}) \mapsto \sigma(\alpha, \vec{d}) \oplus -\sigma(\alpha, \vec{d}). \quad \square$$

**Lemma 9.4.** *For each  $\vec{c} \in \Delta - \Gamma$ , we have the commutative diagram with exact rows:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & D_i^{\vec{c}}(\Delta) & \xrightarrow{\text{id}} & D_i^{\vec{c}}(\Delta) \longrightarrow 0 \\ & & \downarrow 0 & & \downarrow \phi_i^{\Delta} \oplus 0 & & \downarrow \phi_i^{\Delta \cup \Gamma} \\ 0 & \longrightarrow & C_i(Q_{\Delta \cap \Gamma}^*) & \longrightarrow & C_i(Q_{\Delta}^*) \oplus C_i(Q_{\Gamma}^*) & \longrightarrow & C_i(Q_{\Delta \cup \Gamma}^*) \longrightarrow 0 \end{array}$$

*Proof.* The left-hand square is trivially commutative. We show the right-hand square commutes by first tracing right then down:

$$\vec{c} \otimes \pi \mapsto \vec{c} \otimes \pi \mapsto \sigma(\pi, \vec{c}).$$

Now we trace down then right:

$$\vec{c} \otimes \pi \mapsto \sigma(\pi, \vec{c}) \oplus 0 \mapsto \sigma(\pi, \vec{c}) + 0 = \sigma(\pi, \vec{c})$$

Exactness of the rows in the diagram follows from Lemma 9.3, as the bottom row has remained unchanged.  $\square$

**Lemma 9.5.** *The following diagram is commutative, and its rows are exact.*

$$\begin{array}{ccccccc}
0 & \longrightarrow & D_i(\Delta \cap \Gamma) & \longrightarrow & D_i(\Delta) \oplus D_i(\Gamma) & \longrightarrow & D_i(\Delta \cup \Gamma) \longrightarrow 0 \\
& & \downarrow \phi_i^{\Delta \cap \Gamma} & & \downarrow \phi_i^\Delta \oplus \phi_i^\Gamma & & \downarrow \phi_i^{\Delta \cup \Gamma} \\
0 & \longrightarrow & C_i(Q_{\Delta \cap \Gamma}^*) & \longrightarrow & C_i(Q_\Delta^*) \oplus C_i(Q_\Gamma^*) & \longrightarrow & C_i(Q_{\Delta \cup \Gamma}^*) \longrightarrow 0
\end{array}$$

*Proof.* The proof is to take direct sums of the previous two short exact sequences. First, take the direct sum of the diagram in Lemma 9.3 for each  $\vec{c} \in \Delta \cap \Gamma$ . Next, take the resulting short exact sequence of chain complexes and take its direct sum with the diagram in Lemma 9.4 for each  $\vec{c} \in \Delta - \Gamma$ . Finally, switch  $\Delta$  and  $\Gamma$  in Lemma 9.4 and take the direct sum of the resulting diagram with the diagram from Lemma 9.4 for each  $\vec{c} \in \Gamma - \Delta$ . Observe that the second row of the diagram remains the same throughout this process. Also, note that the top row is exact as it is the direct sum of exact sequences. All together, this yields the desired commutative diagram.  $\square$

**Proposition 9.6.** *The following diagram is commutative, and its rows are exact.*

$$\begin{array}{ccccccc}
0 & \longrightarrow & E_i(\Delta \cap \Gamma) & \longrightarrow & E_i(\Delta) \oplus E_i(\Gamma) & \longrightarrow & E_i(\Delta \cup \Gamma) \longrightarrow 0 \\
& & \downarrow \phi_i^{\Delta \cap \Gamma} & & \downarrow \phi_i^\Delta \oplus \phi_i^\Gamma & & \downarrow \phi_i^{\Delta \cup \Gamma} \\
0 & \longrightarrow & C_i(Q_{\Delta \cap \Gamma}^*) & \longrightarrow & C_i(Q_\Delta^*) \oplus C_i(Q_\Gamma^*) & \longrightarrow & C_i(Q_{\Delta \cup \Gamma}^*) \longrightarrow 0
\end{array}$$

*Proof.* Since  $E_i(\Delta)$  is a subspace of  $D_i(\Delta)$ , it follows from Lemma 9.5 that the diagram is commutative. Furthermore, that the second row is exact also follows from this lemma. It remains to show that the first row is exact. However, this follows by the same reasoning that the first row of Lemma 9.5 is exact, but with the permutation module  $M^{B^\#}(\vec{c})$  replaced with the Specht module  $S^{B^*}(\vec{c})$ .  $\square$

**Proposition 9.7.** *Assume that Theorem 8.2 holds for the simplicial complexes  $\Delta$ ,  $\Gamma$ , and the intersection  $\Delta \cap \Gamma$ . Then Theorem 8.2 also holds for the union  $\Delta \cup \Gamma$ .*

*Proof.* Consider the diagram of short exact sequences of chain complexes given in Proposition 9.6. Use the zig-zag lemma to obtain the Mayer–Vietoris sequence:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & K_i(\Delta \cap \Gamma) & \longrightarrow & K_i(\Delta) \oplus K_i(\Gamma) & \longrightarrow & K_i(\Delta \cup \Gamma) \longrightarrow \cdots \\
& & \downarrow \phi_i^{\Delta \cap \Gamma} & & \downarrow \phi_i^\Delta \oplus \phi_i^\Gamma & & \downarrow \phi_i^{\Delta \cup \Gamma} \\
\cdots & \longrightarrow & \tilde{H}_i(Q_{\Delta \cap \Gamma}^*) & \longrightarrow & \tilde{H}_i(Q_\Delta^*) \oplus \tilde{H}_i(Q_\Gamma^*) & \longrightarrow & \tilde{H}_i(Q_{\Delta \cup \Gamma}^*) \longrightarrow \cdots
\end{array}$$

The assumption that Theorem 8.2 holds for the complexes  $\Delta \cap \Gamma$ ,  $\Delta$ , and  $\Gamma$  implies that  $\phi_i^{\Delta \cap \Gamma}$  and  $\phi_i^\Delta \oplus \phi_i^\Gamma$  are isomorphisms. The five-lemma now implies that  $\phi_i^{\Delta \cup \Gamma}$  is also an isomorphism. Furthermore,  $\phi_i^{\Delta \cup \Gamma}$  is an  $\mathfrak{S}_{n-1}$ -equivariant map by Lemma 7.6.  $\square$

*Proof of Theorem 8.2.* Since every simplicial complex  $\Delta$  is the union of simplexes, Proposition 9.7 implies that it is enough to prove Theorem 8.2 for simplexes and the empty simplex. This was done in Proposition 8.3 and Corollary 8.4.  $\square$

## 10. ALTERNATE PROOF OF THEOREM 6.4

As mentioned in the introduction, we now give an alternate proof of Theorem 6.4 using a poset fiber theorem of Björner, Wachs and Welker [3].

**Theorem 10.1.** *Let  $\Delta$  be a simplicial complex of compositions of  $n$ . Then the  $i$ th reduced homology group of the simplicial complex  $Q_\Delta^*$  is given by*

$$\tilde{H}_i(Q_\Delta^*) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in \Delta} \tilde{H}_{i-|\vec{c}|+1}(\text{lk}_{\vec{c}}(\Delta)) \otimes S^{B^*(\vec{c})}.$$

*Proof.* Consider the two posets  $\Delta$  and  $Q_\Delta^*$  with the reverse order  $\leq^*$  and the poset map

$$\text{type} : Q_\Delta^* - \{([n])\} \longrightarrow \Delta - \{(n)\}.$$

Observe that the type map respects the action of the symmetric group  $\mathfrak{S}_{n-1}$ . Now the inverse image  $\text{type}^{-1}(\Delta_{\leq^* \vec{c}})$  is the filter  $Q_{\vec{c}}^*$ . Since  $Q_{\vec{c}}^*$  only has reduced homology in dimension  $|\vec{c}| - 2$  by Theorem 6.3, we have that the fiber  $\Delta(\text{type}^{-1}(\Delta_{\leq^* \vec{c}}))$  is  $(|\vec{c}| - 3)$ -acyclic, where  $|\vec{c}| - 3$  is the length of the longest chain in  $\text{type}^{-1}(\Delta_{\leq^* \vec{c}})$ . Hence Theorem 9.1 of [3] applies. Since  $\mathfrak{S}_{n-1}$  acts trivially on  $\Delta$  (see Remark 6.2), we have that the stabilizer  $\text{Stab}_{\mathfrak{S}_{n-1}}(\vec{c})$  is in fact the whole group  $\mathfrak{S}_{n-1}$ . Thus there is no representation to induce and we have

$$\begin{aligned} \tilde{H}_i(Q_\Delta^*) &\cong_{\mathfrak{S}_{n-1}} \tilde{H}_i(\Delta) \oplus \bigoplus_{\vec{c} \in \Delta - \{(n)\}} \tilde{H}_{|\vec{c}|-2}(\text{type}^{-1}(\Delta_{\leq^* \vec{c}})) \otimes \tilde{H}_{i-|\vec{c}|+1}((\Delta - \{(n)\})_{>^* \vec{c}}) \\ &\cong_{\mathfrak{S}_{n-1}} \tilde{H}_i(\Delta) \oplus \bigoplus_{\vec{c} \in \Delta - \{(n)\}} S^{B^*(\vec{c})} \otimes \tilde{H}_{i-|\vec{c}|+1}(\text{lk}_{\vec{c}}(\Delta)), \end{aligned}$$

where the first summand corresponds to  $\vec{c} = (n)$  and the trivial representation  $S^{B^*(n)}$ , proving the result.  $\square$

## 11. FILTERS IN THE SET PARTITION LATTICE

In Theorem 6.4 we characterized each homology group of  $Q_\Delta^*$ , a subspace of ordered set partitions. We will now translate the topological data we have gathered on  $Q_\Delta^*$  into data on the usual partition lattice  $\Pi_n$ .

Recall that  $Q_\Delta^*$  is the collection of ordered set partitions containing the element  $n$  in the last block, whose type is contained in the simplicial complex  $\Delta \subseteq \text{Comp}(n)$ . Define the *forgetful map*  $f : Q_\Delta^* \longrightarrow \Pi_n$  given by removing the order between blocks, that is,  $f((C_1, C_2, \dots, C_k)) = \{C_1, C_2, \dots, C_k\}$ .

**Definition 11.1.** *Let  $\Pi_\Delta^* \subseteq \Pi_n$  be the image of  $Q_\Delta^*$  under the forgetful map  $f$ .*

**Lemma 11.2.** *Suppose that  $F$  is a filter in the integer partition lattice. Let  $\Delta_F$  be the filter of compositions given by  $\{\vec{c} \in \text{Comp}(n) : \text{type}(\vec{c}) \in F\}$ . Then the associated filter  $\Pi_{\Delta_F}^*$  in the partition lattice is given by  $\{\pi \in \Pi_n : \text{type}(\pi) \in F\}$ .*

*Proof.* Choose  $\pi \in \Pi_n$  such that  $\text{type}(\pi) \in F$ , with  $\pi = \{B_1, B_2, \dots, B_k\}$  where we assume  $n \in B_k$ . The ordered set partition  $\tau = (B_1, B_2, \dots, B_k)$  is an element of  $Q_{\Delta_n}^*$ , since  $\text{type}(\tau) = \text{type}(\pi) \in F$ . Hence  $\pi$  is in the image of the forgetful map  $f$ . The other direction is clear.  $\square$

**Remark 11.3.** In general, taking the image of a filter  $\Delta \subseteq \text{Comp}(n)$  under the map  $\text{type}$  does not define a filter in the integer partition lattice  $I_n$ . For example, consider the simplex  $\Delta$  in  $\text{Comp}(6)$  generated by  $(3, 2, 1)$ . Note that  $\text{type}(\Delta)$  consists of the four partitions  $\{\{3, 2, 1\}, \{3+2, 1\}, \{3, 2+1\}, \{3+2+1\}\} = \{\{3, 2, 1\}, \{5, 1\}, \{3, 3\}, \{6\}\}$ . This is not a filter in  $I_6$  since it does not contain the partition  $\{4, 2\}$ .

**Lemma 11.4.** *The forgetful map  $f : Q_\Delta^* \longrightarrow \Pi_\Delta^*$  respects the  $\mathfrak{S}_{n-1}$ -action.*

*Proof.* Let  $\alpha \in \mathfrak{S}_{n-1}$  and  $\sigma = (C_1, \dots, C_k) \in Q_\Delta^*$ . Then we have that

$$f(\alpha \cdot \sigma) = f((\alpha(C_1), \dots, \alpha(C_k))) = \{\alpha(C_1), \dots, \alpha(C_k)\} = \alpha \cdot f(\sigma). \quad \square$$

The  $\mathfrak{S}_{n-1}$  action on  $\Pi_\Delta^*$  extends to the chains in the order complex  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$ .

For a statement of the equivariant version of the Quillen Fiber Lemma, see [21, Theorem 5.2.2].

**Proposition 11.5.** *The forgetful map  $f : Q_\Delta^* - \{\hat{1}\} \rightarrow \Pi_\Delta^* - \{\hat{1}\} = P$  satisfies the condition of Quillen's Equivariant Fiber Lemma, that is, for a partition  $\pi = \{B_1, B_2, \dots, B_k\}$  in  $P$ , the order complex  $\Delta(f^{-1}(P_{\geq \pi}))$  is the barycentric subdivision of a cone, and is therefore contractible and acyclic.*

*Proof.* Let  $B_k$  be the block of the partition  $\pi$  that contains the element  $n$ . Note that because every ordered partition in  $Q_\Delta^*$  must have the element  $n$  in its last block, we must have that each ordered set partition in the fiber  $f^{-1}(\pi)$  has the set  $B_k$  as its last block. Furthermore, the last block of each ordered set partition in  $f^{-1}(P_{\geq \pi})$  contains the block  $B_k$ .

We claim that  $f^{-1}(P_{\geq \pi})$  is a cone with apex  $([n] - B_k, B_k)$ . Let  $\sigma \in f^{-1}(P_{\geq \pi})$  be the ordered set partition  $\sigma = (C_1, \dots, C_{p-1}, C_p)$ . Note that the number of blocks of  $\sigma$ , is greater than or equal to 2 as we have removed the maximal element  $\hat{1}$  from  $Q_\Delta^*$ . If  $C_p = B_k$  then the face  $\sigma$  contains the vertex  $([n] - B_k, B_k)$ . If  $C_p \supsetneq B_k$  then both  $\sigma$  and the vertex  $([n] - B_k, B_k)$  are contained in the face  $(C_1, \dots, C_{p-1}, C_p - B_k, B_k)$  in  $f^{-1}(P_{\geq \pi})$ . Thus  $f^{-1}(P_{\geq \pi})$  is the face poset of a cone with vertex  $([n] - B_k, B_k)$ , and therefore  $\Delta(f^{-1}(P_{\geq \pi}))$  is the barycentric subdivision of a cone and hence contractible and acyclic.  $\square$

Combining Proposition 11.5 with Theorem 6.4, we have the following result for the homology of the order complex  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$ .

**Theorem 11.6.** *The  $i$ th reduced homology group of the order complex of  $\Pi_\Delta^* - \{\hat{1}\}$  as an  $\mathfrak{S}_{n-1}$ -module is given by*

$$\tilde{H}_i(\Delta(\Pi_\Delta^* - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in \Delta} \tilde{H}_{i-|\vec{c}|+1}(\text{lk}_{\vec{c}}(\Delta)) \otimes S^{B^*(\vec{c})}.$$

**Remark 11.7.** Suppose that  $\text{lk}_{\vec{c}}(\Delta)$  has reduced homology in dimension  $j$ . By Theorem 11.6 this reduced homology contributes to dimension  $j + |\vec{c}| - 1$  of the reduced homology of the order complex of  $\Pi_\Delta^* - \{\hat{1}\}$ .

We end the section with a discussion of Morse matchings in the link  $\text{lk}_{\vec{c}}(\Delta)$ . Assume that the link  $\text{lk}_{\vec{c}}(\Delta)$  has a discrete Morse matching with critical cell  $\vec{d}$ , which also contributes to the reduced homology of  $\text{lk}_{\vec{c}}(\Delta)$ . For instance, this case occurs if  $\vec{d}$  is a facet. Similarly,  $\vec{d}$  will contribute to the reduced homology of  $\text{lk}_{\vec{c}}(\Delta)$  if  $\vec{d}$  is a homology facet of a shelling. In either case, the critical cell  $\vec{d}$  contributes to the reduced homology of  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  in dimension  $\dim_{\text{lk}_{\vec{c}}(\Delta)}(\vec{d}) + |\vec{c}| - 1 = \dim_\Delta(\vec{d}) = |\vec{d}| - 2$ , by equation (6.1). Note that this dimension is independent of the composition  $\vec{c}$ .

## 12. CONSEQUENCES OF THE MAIN RESULT

As the title of this section suggests, we will now derive results from Theorem 11.6 using topological data from  $\Delta$ .

**Theorem 12.1.** *Assume that  $\Delta$  is homeomorphic to a  $k$ -dimensional manifold with or without boundary. Then the reduced homology of the order complex  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  is given by*

$$\tilde{H}_i(\Delta(\Pi_\Delta^* - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \tilde{H}_i(\Delta) \otimes 1_{\mathfrak{S}_{n-1}} \quad \text{for } i < k,$$

and the top dimensional homology is given by

$$\tilde{H}_k(\Delta(\Pi_\Delta^* - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \tilde{H}_k(\Delta) \otimes 1_{\mathfrak{S}_{n-1}} \oplus \bigoplus_{\vec{c} \in \text{Int}(\Delta)} S^{B^*(\vec{c})},$$

where  $1_{\mathfrak{S}_{n-1}}$  is the trivial representation of  $\mathfrak{S}_{n-1}$ , and the direct sum is over the interior faces of the manifold  $\Delta$ . Moreover, these isomorphisms hold as  $\mathfrak{S}_{n-1}$ -modules.

*Proof.* Since  $\Delta$  is homeomorphic to a  $k$ -dimensional manifold, we may apply the comment preceding Proposition 3.8.9 of [16], which states that for any  $\vec{c} \in \Delta$ , where  $\vec{c}$  is not the empty composition  $(n)$ , we have that  $\text{lk}_{\vec{c}}(\Delta)$  has the homology groups of a sphere of dimension  $k - |\vec{c}| + 1$  if  $\vec{c}$  is on the interior of  $\Delta$ , or the homology groups of a ball of dimension  $k - |\vec{c}| + 1$  if  $\vec{c}$  is on the boundary. Hence if  $\vec{c}$  is on the boundary of  $\Delta$  it does not contribute to the reduced homology of  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$ . If instead  $\vec{c}$  is in the interior of  $\Delta$  then by Remark 11.7 it will contribute to the reduced homology group of dimension  $(k - |\vec{c}| + 1) + |\vec{c}| - 1 = k$ . This is the top homology of the complex. Finally, observe that the composition  $(n)$  contributes to all homology groups of  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  when  $\Delta$  has nontrivial homology, and that the Specht module  $S^{B^*(n)}$  is the trivial representation  $1_{\mathfrak{S}_{n-1}}$ .  $\square$

We now give two immediate corollaries of Theorem 12.1, when  $\Delta$  is homeomorphic to a sphere or a ball.

**Corollary 12.2.** *Suppose that  $\Delta$  is homeomorphic to a sphere of dimension  $k$ . Then the order complex  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  only has homology in dimension  $k$  given by*

$$\tilde{H}_k(\Delta(\Pi_\Delta^* - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in \Delta} S^{B^*(\vec{c})}.$$

**Corollary 12.3.** *Suppose that  $\Delta$  is homeomorphic to a ball of dimension  $k$ . Then the order complex  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  only has homology in dimension  $k$  given by*

$$\tilde{H}_k(\Delta(\Pi_\Delta^* - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in \text{Int}(\Delta)} S^{B^*(\vec{c})}.$$

Next we obtain a result about  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  when  $\Delta$  is shellable.

**Proposition 12.4.** *Suppose that  $\Delta$  is a shellable complex of dimension  $k$ . Then the order complex  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  only has reduced homology in dimension  $k$  given by*

$$\tilde{H}_k(\Delta(\Pi_\Delta^* - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in \Delta} \tilde{\beta}_{k-|\vec{c}|+1}(\text{lk}_{\vec{c}}(\Delta)) \cdot S^{B^*(\vec{c})}.$$

*Proof.* Note that the face  $\vec{c}$  has dimension  $|\vec{c}| - 2$ . Hence the link  $\text{lk}_{\vec{c}}(\Delta)$  has dimension  $k - \dim(\vec{c}) - 1 = k - |\vec{c}| + 1$ , by equation (6.1). Since the link is shellable, all of its reduced homology occurs in dimension  $k - |\vec{c}| + 1$  and this contributes only to the reduced homology of dimension  $k$  of  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  by Remark 11.7. Lastly, the betti number  $\tilde{\beta}_{k-|\vec{c}|+1}$  is explained by the fact that a shellable complex has the homotopy type of a wedge of spheres of the same dimension.  $\square$

## 13. THE REPRESENTATION RING

The *representation ring*  $R(G)$  of a group  $G$  is the free abelian group with generators given by representations  $V$  of  $G$  modulo the subgroup generated by  $V + W - V \oplus W$ . Elements of the representation ring are called virtual representations because summands can have negative coefficients. For finite groups, complete reducibility implies  $R(G)$  is just the free abelian group generated by the irreducible representations  $V$  of  $G$ .

**Remark 13.1.** Suppose  $G$  acts trivially on the space  $V$ . Then  $V \otimes W \cong_G \dim(V) \cdot W$  in the representation ring  $R(G)$ .

*Proof.* Since  $G$  acts trivially on  $V$  we know that  $V \cong_G \mathbb{C}^{\dim(V)}$ . Thus,  $V \otimes W \cong_G \mathbb{C}^{\dim(V)} \otimes W \cong_G \dim(V) \cdot W$ .  $\square$

In the representation ring we can compute the alternating sum of the homology groups of  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$ , which we do in the following proposition. This can be seen as  $\mathfrak{S}_{n-1}$ -analogue of the reduced Euler characteristic.

**Proposition 13.2.** *As virtual  $\mathfrak{S}_{n-1}$ -representations we have that*

$$\bigoplus_{i \geq -1} (-1)^i \cdot \tilde{H}_i(\Delta(\Pi_\Delta^* - \{\hat{1}\})) \cong \bigoplus_{\vec{c} \in \Delta} (-1)^{|\vec{c}|-1} \cdot \tilde{\chi}(\text{lk}_{\vec{c}}(\Delta)) \cdot S^{B^*(\vec{c})}.$$

*Proof.* We begin the proof by applying alternating sums to both sides of Theorem 11.6.

$$\begin{aligned} \bigoplus_{i \geq -1} (-1)^i \cdot \tilde{H}_i(\Delta(\Pi_\Delta^* - \{\hat{1}\})) &\cong \bigoplus_{i \geq -1} (-1)^i \cdot \bigoplus_{\vec{c} \in \Delta} \tilde{H}_{i-|\vec{c}|+1}(\text{lk}_{\vec{c}}(\Delta)) \otimes S^{B^*(\vec{c})} \\ &\cong \bigoplus_{\vec{c} \in \Delta} \bigoplus_{i \geq -1} (-1)^i \cdot \tilde{\beta}_{i-|\vec{c}|+1}(\text{lk}_{\vec{c}}(\Delta)) \cdot S^{B^*(\vec{c})} \\ &\cong \bigoplus_{\vec{c} \in \Delta} (-1)^{|\vec{c}|-1} \cdot \bigoplus_{j \geq -1} (-1)^j \cdot \tilde{\beta}_j(\text{lk}_{\vec{c}}(\Delta)) \cdot S^{B^*(\vec{c})} \\ &\cong \bigoplus_{\vec{c} \in \Delta} (-1)^{|\vec{c}|-1} \cdot \tilde{\chi}(\text{lk}_{\vec{c}}(\Delta)) \cdot S^{B^*(\vec{c})}, \end{aligned}$$

where the second step is by Remark 13.1, since  $\mathfrak{S}_{n-1}$  acts trivially on  $\tilde{H}_{i-|\vec{c}|+1}(\text{lk}_{\vec{c}}(\Delta))$ . In the last step we used that the alternating sum of the Betti numbers is the reduced Euler characteristic.  $\square$

The next lemma is straightforward to prove using jeu-de-taquin; see [13] or [15, A.1.2].

**Lemma 13.3.** *The permutation module  $M^{B^\#(\vec{c})}$  is isomorphic to the direct sum over all border strip Specht modules  $S^{B^*(\vec{d})}$  for  $\vec{d} \leq^* \vec{c}$ , that is,*

$$M^{B^\#(\vec{c})} \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{d} \leq^* \vec{c}} S^{B^*(\vec{d})}.$$

*Proof.* Recall that the border strip of shape  $A^\#(\vec{c})$  was defined in Section 5.

We have the isomorphism  $S^{A^\#(\vec{c})} \cong_{\mathfrak{S}_{n-1}} M^{A^\#(\vec{c})}$  because the rows of the shape  $A(\vec{c}/1)$  are non-overlapping, thus polytabloids of shape  $A(\vec{c}/1)$  are tabloids of shape  $A(\vec{c}/1)$ . Additionally, we have  $M^{A^\#(\vec{c})} \cong_{\mathfrak{S}_{n-1}} M^{B^\#(\vec{c})}$ , since tabloids are defined as row equivalence classes of tableaux and  $A(\vec{c}/1)$  and  $B(\vec{c}/1)$  have the same rows. Combining these two  $\mathfrak{S}_{n-1}$ -isomorphisms yields  $M^{B^\#(\vec{c})} \cong S^{A^\#(\vec{c})}$ .



Now consider the  $k-1$  empty boxes situated to the left of every row in the Specht module defined by the shape  $A^\#(\vec{c})$ , but above the last box of the previous row. For each of these boxes perform a jeu-de-taquin slide into this box.

For each slide, there are two alternatives. If the slide is horizontal, it moves the upper row one step to the left such that the two rows overlap in one position. If the slide is vertical then every entry in the lower row moves one step up.

After performing all the  $k-1$  slides the result is a border shape of shape  $B^\#(\vec{c})$ , where the composition  $\vec{c}$  is less than or equal to the composition  $\vec{d}$  in the dual order.  $\square$

Proposition 13.2 can also be proved using the Hopf trace formula; see [21, Theorem 2.3.9].

*Second proof of Proposition 13.2.* Recall that  $\tilde{H}_i(\Delta(\Pi_\Delta^* - \{\hat{1}\})) \cong \tilde{H}_i(Q_\Delta^*)$ . By applying the Hopf trace formula we have that

$$\begin{aligned}
 \bigoplus_{i \geq -1} (-1)^i \cdot \tilde{H}_i(Q_\Delta^*) &\cong \bigoplus_{i \geq -1} (-1)^i \cdot C_i(Q_\Delta^*) \\
 &\cong \bigoplus_{\vec{d} \in \Delta} (-1)^{|\vec{d}|} \cdot M^{B^\#(\vec{d})} \\
 &\cong \bigoplus_{\vec{d} \in \Delta} (-1)^{|\vec{d}|} \cdot \bigoplus_{\vec{c} \leq^* \vec{d}} S^{B^*(\vec{c})} \\
 &\cong \bigoplus_{\vec{c} \in \Delta} \sum_{\substack{\vec{d} \geq^* \vec{c} \\ \vec{d} \in \Delta}} (-1)^{|\vec{d}|} \cdot S^{B^*(\vec{c})} \\
 &\cong \bigoplus_{\vec{c} \in \Delta} (-1)^{|\vec{c}|-1} \cdot \sum_{\substack{\vec{d} \geq^* \vec{c} \\ \vec{d} \in \Delta}} (-1)^{|\vec{d}|-|\vec{c}|-1} \cdot S^{B^*(\vec{c})}.
 \end{aligned}$$

Notice that in the second isomorphism we have used that the chain space  $C_i(Q_\Delta^*)$  has basis given by all ordered set partitions into  $i+2$  parts with type in  $\Delta$ . This is equivalent to the direct sum over all permutation modules  $M^{B^\#(\vec{d})}$  where  $\vec{d} \in \Delta$  is a composition of  $n$  into  $i+2$  parts. The remaining step is to observe that the inner sum of the last line is given by the reduced Euler characteristic  $\tilde{\chi}(\text{lk}_{\vec{c}}(\Delta))$ .  $\square$

We observe that in the case when the order complex  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  has all its reduced homology concentrated in one dimension, the second proof of Proposition 13.2 which uses the Hopf trace formula gives a shorter proof of our main result Theorem 11.6.

Lastly, by taking dimension on both sides of Proposition 13.2 we obtain the reduced Euler characteristic of  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$ .

**Corollary 13.4.** *The reduced Euler characteristic of  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  is given by*

$$\tilde{\chi}(\Delta(\Pi_\Delta^* - \{\hat{1}\})) = \sum_{\vec{c} \in \Delta} (-1)^{|\vec{c}|-1} \cdot \tilde{\chi}(\text{lk}_{\vec{c}}(\Delta)) \cdot \beta_n^*(\vec{c}).$$

This corollary extends Theorem 3.1 from [8].

14. THE HOMOTOPY TYPE OF  $\Pi_\Delta^*$ 

We turn our attention to the homotopy type of the order complex  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$ . By combining the poset fiber theorems of Quillen [12] and Björner, Wachs and Welker [3] we obtain the next result. Recall that  $*$  denotes the (free) join of complexes.

**Theorem 14.1.** *The order complex of  $\Pi_\Delta^* - \{\hat{1}\}$  is homotopy equivalent to the complex of ordered set partitions  $Q_\Delta^*$ , that is,  $\Delta(\Pi_\Delta^* - \{\hat{1}\}) \simeq Q_\Delta^*$ . Furthermore, the following homotopy equivalence holds:*

$$Q_\Delta^* \simeq \Delta(\Delta - \{(n)\}) \vee \{Q_{\vec{c}}^* * \text{lk}_{\vec{c}}(\Delta) : \vec{c} \in \Delta - \{(n)\}\},$$

where  $\vee$  denotes identifying each vertex  $\vec{c}$  in  $\Delta(\Delta - \{(n)\})$  with any vertex in  $Q_{\vec{c}}^*$ . In the case when the complex  $\Delta$  is connected then the homotopy equivalence simplifies to

$$Q_\Delta^* \simeq \bigvee_{\vec{c} \in \Delta} Q_{\vec{c}}^* * \text{lk}_{\vec{c}}(\Delta).$$

*Proof.* The first homotopy equivalence follows by applying Quillen's fiber lemma to the forgetful map  $f$ . This yields  $\Delta(\Pi_\Delta^* - \{\hat{1}\}) \simeq \Delta(Q_\Delta^* - \{\hat{1}\}) \cong Q_\Delta^*$ , since  $\Delta(Q_\Delta^* - \{\hat{1}\})$  is the barycentric subdivision of  $Q_\Delta^*$ .

The second homotopy equivalence in both cases follows by Theorem 1.1 in [3], with the same reasoning as in the proof of Theorem 6.4. Furthermore, when  $\vec{c} = (n)$  then the complex  $Q_{\vec{c}}^*$  is the empty complex, which is the identity for the join.  $\square$

**Corollary 14.2.** *Let  $\Delta$  be a connected simplicial complex. Assume furthermore that each link (including  $\Delta$ )  $\text{lk}_{\vec{c}}(\Delta)$  is a wedge of spheres. Then the order complex  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  is also a wedge of spheres. Furthermore, the number of  $i$ -dimensional spheres is given by the sum*

$$(14.1) \quad \sum_{\vec{c} \in \Delta} \beta_n^*(\vec{c}) \cdot \tilde{\beta}_{i-|\vec{c}|+1}(\text{lk}_{\vec{c}}(\Delta)),$$

where  $\tilde{\beta}_j$  denotes the  $j$ th reduced Betti number.

Next we have the homotopy versions of Corollaries 12.2 and 12.3. To prove the next two corollaries, we are again using the comment preceding Proposition 3.8.9 of [16] to determine the reduced Betti numbers of the links.

**Corollary 14.3.** *Suppose that  $\Delta$  is homeomorphic to a sphere of dimension  $k$ . Then the order complex  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  is a wedge of  $k$ -dimensional spheres and the number of spheres is given by the sum:*

$$\sum_{\vec{c} \in \Delta} \beta_n^*(\vec{c}).$$

**Corollary 14.4.** *Suppose that  $\Delta$  is homeomorphic to a ball of dimension  $k$ . Then the order complex  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  is a wedge of  $k$ -dimensional spheres and the number of spheres is given by the sum:*

$$\sum_{\vec{c} \in \text{Int}(\Delta)} \beta_n^*(\vec{c}).$$

We end this section with a discussion of how we can lift discrete Morse matchings from the links of  $\Delta$  to the complex of order set partitions  $Q_\Delta^*$ .

**Definition 14.5.** *For an ordered set partition  $\sigma = (C_1, C_2, \dots, C_k)$  of  $n$ , where  $C_i = \{c_{i,1} < c_{i,2} < \dots < c_{i,j_i}\}$  and  $|C_i| = j_i$ , define the permutation  $\text{perm}(\sigma) \in \mathfrak{S}_n$  to be the elements of the blocks listed in the order of the blocks, that is,*

$$\text{perm}(\sigma) = c_{1,1}, c_{1,2}, \dots, c_{1,j_1}, c_{2,1}, c_{2,2}, \dots, c_{k,j_k}.$$

Define the descent set of an ordered set partition  $\sigma$  to be  $\text{Des}(\sigma) = \text{Des}(\text{perm}(\sigma))$ . Observe that the descent composition of an ordered set partition is an order preserving map from the poset of ordered set partitions  $Q_n$  to the poset of compositions  $\text{Comp}_n$ , that is,  $\text{Des} : Q_n^* \rightarrow \text{Comp}(n)$  is a poset map.

**Lemma 14.6.** *Let  $\Delta$  be a filter in the composition poset  $\text{Comp}(n)$ . For the order preserving map  $\text{Des} : Q_\Delta^* \rightarrow \Delta$  the poset fiber  $\text{Des}^{-1}(\vec{c})$  is the (poset) direct sum of  $\beta_n^*(\vec{c})$  copies of the poset  $\text{lk}_{\vec{c}}(\Delta) = \{\vec{d} \in \Delta : \vec{d} \leq^* \vec{c}\}$ .*

*Proof.* Let  $\sigma$  be an ordered set partition and assume that the  $i$ th block  $C_i$  is the disjoint union of the two non-empty sets  $X$  and  $Y$  such that  $\max(X) < \min(Y)$ . Observe now that the two ordered set partitions  $\sigma$  and  $(\dots, C_{i-1}, X, Y, C_{i+1}, \dots)$  have the same descent composition, since there is no descent between blocks  $X$  and  $Y$ .

Let  $\vec{c}$  be a composition in the filter  $\Delta$ . For any ordered set partition  $\tau$  in the fiber  $\text{Des}^{-1}(\vec{c})$  we know that  $\tau$  has descent composition  $\vec{c}$ , that is,  $\text{Des}(\tau) = \vec{c}$ . As  $\tau$  can only have descents between blocks, we know the minimal elements of  $\text{Des}^{-1}(\vec{c})$  have the form  $\sigma(\alpha, \vec{c})$  for  $\alpha \in \mathfrak{S}_n$  satisfying  $\text{Des}(\alpha) = \vec{c}$  and  $\alpha_n = n$ . To remain in the same fiber as these minimal elements, we are free to break blocks as in previous paragraph, hence

$$\text{Des}^{-1}(\vec{c}) = \{\sigma(\alpha, \vec{d}) : \vec{c} \leq^* \vec{d}, \vec{d} \in \Delta, \text{Des}(\alpha) = \vec{c}, \alpha_n = n\}.$$

Notice that the poset  $\text{lk}_{\vec{c}}(\Delta)$  is isomorphic to the poset  $\{\sigma(\alpha, \vec{d}) : \vec{d} \leq \vec{c}, \vec{d} \in \Delta\}$  for a fixed permutation  $\alpha \in \mathfrak{S}_n$  satisfying  $\text{Des}(\alpha) = \vec{c}$  and  $\alpha_n = n$ . Finally, for a composition  $\vec{c} \in \text{lk}_{\vec{c}}(\Delta)$  and a permutation  $\beta \in \mathfrak{S}_n$  different from  $\alpha$  such that  $\text{Des}(\beta) = \vec{c}$  and  $\beta_n = n$ , consider the two ordered set partitions  $\sigma(\beta, \vec{c})$  and  $\sigma(\alpha, \vec{d})$ , where  $\vec{d} \in \text{lk}_{\vec{c}}(\Delta)$ . By examining the first increasing run in the permutations  $\alpha$  and  $\beta$  where their elements differ, we conclude that the two ordered set partitions  $\sigma(\beta, \vec{c})$  and  $\sigma(\alpha, \vec{d})$  are incomparable. Thus the fiber  $\text{Des}^{-1}(\vec{c})$  is a direct sum of copies of the poset  $\text{lk}_{\vec{c}}(\Delta)$ , one for each permutation  $\alpha$  in  $\mathfrak{S}_n$  satisfying  $\text{Des}(\alpha) = \vec{c}$  and  $\alpha(n) = n$ .  $\square$

**Theorem 14.7.** *Let  $\Delta$  be a simplicial complex of compositions such that every link  $\text{lk}_{\vec{c}}(\Delta)$  has a Morse matching where the critical cells are facets of the link  $\text{lk}_{\vec{c}}(\Delta)$ . Then the simplicial complex  $Q_\Delta^*$  has a Morse matching, where the number of  $i$ -dimensional critical cells is given by equation (14.1).*

*Proof.* Apply the Patchwork Theorem [9, Theorem 11.10] to the poset map  $\text{Des} : Q_\Delta^* \rightarrow \Delta$ . By Lemma 14.6, each fiber is a direct sum of links of  $\Delta$ , each of which has a Morse matching. Each critical cell is a facet. Hence  $Q_\Delta^*$  is homotopy equivalent to a wedge of spheres, and thus the order complex  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  is also a wedge of spheres. The number of  $i$ -dimensional critical cells of  $Q_\Delta^*$  in the fiber  $\text{Des}^{-1}(\vec{c})$  is the number of critical cells of dimension  $i - |\vec{c}| + 1$  in the link  $\text{lk}_{\vec{c}}(\Delta)$  times the number of copies of the link, that is  $\beta_n^*(\vec{c})$ . By summing over all compositions  $\vec{c}$  in  $\Delta$  the result follows.  $\square$

Now suppose that  $\Delta$  is a non-pure shellable complex in the sense of [2]. Then each link in  $\Delta$  is also shellable, and thus for each link there exists a discrete Morse matching whose critical cells are facets of the link; see Chapter 12 of [9].

**Corollary 14.8.** *If  $\Delta$  is a non-pure shellable complex then Theorem 14.7 applies and the simplicial complex  $Q_\Delta^*$  has a Morse matching where the number of  $i$ -dimensional critical cells is given by equation (14.1).*

*Proof.* This follows directly from two observations: (i) a non-pure shellable complex has a Morse matching with all critical cells being facets, (ii) each link of a non-pure shellable complex is non-pure shellable. See Section 12.1 in [9].  $\square$

## 15. EXAMPLES

In this section we use Theorem 11.6 and its consequences from Section 12 to derive results about various filters  $\Pi_\Delta^*$ .

**Example 15.1.** Let  $\vec{d}$  be a composition of  $n$  into  $k + 2$  parts and let  $\Delta$  be the simplex generated by  $\vec{d}$ . Since the simplex is homeomorphic to a  $k$ -dimensional ball, by Corollary 12.3 we have that the  $k$ th reduced homology group is given by

$$\tilde{H}_k(\Delta(\Pi_\Delta^* - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} S^{B^*(\vec{d})},$$

since the only face of  $\Delta$  in the interior of  $\Delta$  is the facet  $\vec{d}$ . This example illustrates Theorems 5.3 and 7.4 in [7]. Moreover, this is the base case of the authors' proof of Theorem 6.4 using the Mayer–Vietoris sequence.

**Example 15.2.** Let  $\vec{d}$  be a composition into  $k + 3$  parts and let  $\Delta$  be the boundary of the simplex generated by  $\vec{d}$ , that is,  $\Delta$  is homeomorphic to a  $k$ -dimensional sphere. Then  $\Delta$  is shellable and the order complex  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  is a wedge of  $k$ -dimensional spheres. Now by Corollary 12.2 we have that the  $k$ th reduced homology group is given by

$$\tilde{H}_k(\Delta(\Pi_\Delta^* - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} <^* \vec{d}} S^{B^*(\vec{c})} \cong_{\mathfrak{S}_{n-1}} M^{B^\#(\vec{d})} / S^{B^*(\vec{d})}.$$

Note that we have used Lemma 13.3 to express the permutation module  $M^{B^\#(\vec{d})}$  as a direct sum of Specht modules.

**Example 15.3.** Let  $\vec{d}$  be a composition of  $n$  into  $k + r$  parts, where  $r \geq 1$ . Let  $\Delta$  be the  $k$ -skeleton of the simplex generated by the composition  $\vec{d}$ . Note that  $\Delta$  is shellable, so by Corollary 14.8 the order complex  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  is a wedge of  $k$ -dimensional spheres. By Proposition 12.4 we have the following calculation in the representation ring:

$$(15.1) \quad \tilde{H}_k(\Delta(\Pi_\Delta^* - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\substack{\vec{c} <^* \vec{d} \\ |\vec{c}| \leq k+2}} \binom{k+r-|\vec{c}|-1}{k-|\vec{c}|+2} \cdot S^{B^*(\vec{c})}.$$

Here we have used that  $\tilde{\beta}_{k-|\vec{c}|+1}(\text{lk}_{\vec{c}}(\Delta)) = (-1)^{k-|\vec{c}|+1} \cdot \tilde{\chi}(\text{lk}_{\vec{c}}(\Delta))$  since  $\text{lk}_{\vec{c}}(\Delta)$  is shellable. Lastly, we also used a basic identity on the alternating sum of binomial coefficients, which arises in computing the Euler characteristic of the link.

**Example 15.4** (The  $d$ -divisible partition lattice with minimal elements removed). Let  $n$  be a multiple of  $d$ . Consider the boundary of the simplex generated by the composition  $(d, d, \dots, d)$  of  $n$ . Then  $\Delta$  is a  $(n/d - 3)$ -dimensional simplicial complex, and  $\Pi_\Delta^*$  is the  $d$ -divisible partition lattice without its minimal elements. By applying Example 15.2 we obtain that  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  is a wedge of  $(n/d - 3)$ -dimensional spheres and the reduced homology group is given by  $\tilde{H}_{n/d-3}(\Pi_\Delta^* - \{\hat{1}\}) \cong_{\mathfrak{S}_{n-1}} M^{B^\#(d, \dots, d, d)} / S^{B^*(d, d, \dots, d)}$ .

Setting  $d = 1$  in the last example shows that the action of  $\mathfrak{S}_{n-1}$  on the reduced homology group of  $\Delta(\Pi_n - \{\hat{0}, \hat{1}\})$  is  $M^{B^\#(1, \dots, 1, 1)} = M^{B(1, \dots, 1)}$ , which is the regular representation of  $\mathfrak{S}_{n-1}$ .

**Example 15.5** (The truncated  $d$ -divisible partition lattice). To generalize Example 15.4 and specialize Example 15.3, let  $n = (k + r) \cdot d$  and consider the  $k$ -skeleton of the simplex generated by the composition  $(d, d, \dots, d)$  of  $n$ . Here  $\Pi_\Delta^*$  consists of all set partitions in the  $d$ -divisible partition lattice with at most  $k + 2$  parts. Directly we have that the order complex  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  is a wedge of  $k$ -dimensional spheres and its  $k$ -dimensional reduced homology is given by equation (15.1).

**Example 15.6.** An integer partition  $\lambda = \{\lambda_1^{m_1}, \lambda_2^{m_2}, \dots, \lambda_p^{m_p}\}$  of the non-negative integer  $n$  is called a *knapsack partition* if all the sums  $\sum_{i=1}^p e_i \cdot \lambda_i$ , where  $0 \leq e_i \leq m_i$ , are distinct. In other words,  $\lambda$  is a knapsack partition if

$$\left| \left\{ \sum_{i=1}^p j_i \cdot \lambda_i : 0 \leq j_i \leq m_i \right\} \right| = \prod_{i=1}^p (m_i + 1).$$

For a knapsack partition  $\lambda$  into  $k-1$  parts of  $n-m$ , where  $m < n$ , define  $\Delta_{\lambda,m}$  to be the simplicial complex  $\Delta_{\lambda,m}$  which has the facets  $(c_1, c_2, \dots, c_{k-1}, c_k)$  with  $\text{type}(c_1, c_2, \dots, c_{k-1}) = \lambda$  and the last part  $c_k$  is  $m$ . The complex  $\Delta_{\lambda,m}$  is homeomorphic to a  $(k-2)$ -dimensional ball; see the proof of Theorem 4.4 in [8]. Applying Corollary 12.3, we obtain the following result:

$$\tilde{H}_{k-2}(\Delta(\Pi_{\Delta_{\lambda,m}}^* - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in \text{Int}(\Delta_{\lambda,m})} S^{B^*(\vec{c})}.$$

Furthermore, the set of interior faces of  $\Delta_{\lambda,m}$  is given by compositions  $\vec{c}$  in  $\Delta_{\lambda,m}$  such that when each part of  $\vec{c}$  is written as a sum of parts of  $\lambda$ , those parts are distinct. This example is Theorem 10.3 in [7]. Moreover,  $\Delta_{\lambda,m}$  is shellable, so Theorem 14.7 yields a Morse matching of  $Q_{\Delta_{\lambda,m}}^*$ ; see Theorem 8.2 of [7].

## 16. THE FROBENIUS COMPLEX

We now consider a different class of examples stemming from [6]. Let  $\Lambda$  be a semigroup of positive integers, that is, a subset of the positive integers which is closed under addition. Let  $\Delta_n$  be the collection of all compositions of  $n$  whose parts belong to  $\Lambda$ , that is,

$$\Delta_n = \{(c_1, \dots, c_k) \in \text{Comp}(n) : c_1, \dots, c_k \in \Lambda\}.$$

Since  $\Lambda$  is closed under addition, we obtain that  $\Delta_n$  is a filter in the poset of compositions  $\text{Comp}(n)$  and hence we view it as a simplicial complex. This complex is known as the Frobenius complex; see [6]. Moreover, since  $\Lambda$  is a semigroup, the collection of integer partitions of  $n$  with parts in  $\Lambda$  is a filter, therefore, using Lemma 11.2 the associated filter in the partition lattice is given by

$$\Pi_n^\Lambda = \{\{B_1, \dots, B_k\} \in \Pi_n : |B_1|, \dots, |B_k| \in \Lambda\}.$$

Let  $\Psi_n$  be the generating function

$$\Psi_n = \sum_{i \geq -1} \tilde{\beta}_i(\Delta_n) \cdot t^{i+1}.$$

Observe that for a composition  $\vec{c}$  in  $\Delta_n$  we have that the link  $\text{lk}_{\vec{c}}(\Delta_n)$  is given by the join

$$(16.1) \quad \text{lk}_{\vec{c}}(\Delta_n) = \Delta_{c_1} * \Delta_{c_2} * \dots * \Delta_{c_k}.$$

We can apply the Künneth theorem to obtain that the  $i$ th reduced Betti number of the link is given by

$$\tilde{\beta}_i(\text{lk}_{\vec{c}}(\Delta_n)) = [t^{i+1}] \Psi_{c_1} \cdot \Psi_{c_2} \dots \Psi_{c_k}.$$

Using Theorem 11.6, the  $i$ th reduced Betti number of the order complex  $\Delta(\Pi_n^\Lambda - \{\hat{1}\})$  is given in the representation ring of  $\mathfrak{S}_{n-1}$  by

$$\tilde{H}_i(\Delta(\Pi_n^\Lambda - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \sum_{\vec{c}} [t^{i+1}] \Psi_{c_1} \cdot \Psi_{c_2} \dots \Psi_{c_k} \cdot S^{B^*(\vec{c})},$$

where the sum is over all compositions  $\vec{c} = (c_1, c_2, \dots, c_k)$  of  $n$ .

A more explicit approach is possible when the complex  $\Delta_n$  has a discrete Morse matching. By combining equation (16.1), Lemma 4.1, and a Morse matching from [6], we create a Morse matching on every link. We will see this method in the remainder of this section.

We continue by studying one concrete example. Let  $a$  and  $d$  be two positive integers. Let  $\Lambda$  be the semigroup generated by the arithmetic progression

$$\Lambda = \langle a, a + d, a + 2d, \dots \rangle.$$

Since for  $j \geq a$  we have that  $a + j \cdot d = d \cdot a + a + (j - a) \cdot d$ , the semigroup is generated by the finite arithmetic progression

$$\Lambda = \langle a, a + d, a + 2d, \dots, a + (a - 1)d \rangle.$$

Clark and Ehrenborg proved that the Frobenius complex  $\Delta_n$  is a wedge of spheres of different dimensions; see [6, Theorem 5.1]. Observe that their result is formulated in terms of sets, instead of compositions. However, the two notions are equivalent via the natural bijection given by sending a composition  $(c_1, c_2, \dots, c_k)$  of  $n$  to the subset  $\{c_1, c_1 + c_2, \dots, c_1 + \dots + c_{k-1}\}$  of the set  $[n - 1]$ . To state their result, let  $A$  be the set  $\{a + d, a + 2d, \dots, a + (a - 1)d\}$ .

**Proposition 16.1.** *For  $n$  in the semigroup  $\Lambda$ , there is a discrete Morse matching on the Frobenius complex  $\Delta_n$  such that the critical cells are compositions  $\vec{c} = (c_1, \dots, c_k)$  characterized by*

- (i) *All but the last entry of the composition belongs to the set  $A$ , that is,  $c_1, \dots, c_{k-1} \in A$ .*
- (ii) *The last entry  $c_k$  belongs to  $\{a\} \cup A$ .*

*Furthermore, all the critical cells are facets.*

*Proof.* When  $a$  and  $d$  are relative prime, that is,  $\gcd(a, d) = 1$ , this result is Lemma 5.10 in [6]. When  $a$  and  $d$  are not relative prime, the result follows by scaling down the three parameters  $a$ ,  $d$  and  $n$  by  $a' = a/\gcd(a, d)$ ,  $d' = d/\gcd(a, d)$  and  $n' = n/\gcd(a, d)$ . Now the result applies the semigroup  $\Lambda' = \langle a', a' + d', a' + 2d', \dots \rangle$  and its associated Frobenius complex  $\Delta'_{n'}$ . However, this complex is isomorphic to  $\Delta_n$  by sending the composition  $\vec{c} = (c_1, \dots, c_k)$  in  $\Delta'_{n'}$  to the composition  $\gcd(a, d) \cdot \vec{c} = (\gcd(a, d) \cdot c_1, \dots, \gcd(a, d) \cdot c_k)$  in  $\Delta_n$ .  $\square$

**Corollary 16.2.** *The order complex  $\Delta(\Pi_n^\Lambda - \{\hat{1}\})$  is a wedge of spheres.*

*Proof.* Since  $\Delta_n$  has a discrete Morse matching where each critical cell is a facet,  $\Delta_n$  is homotopy equivalent to a wedge of spheres. Furthermore, by equation (16.1) we know that every link of  $\Delta_n$  is a wedge of spheres. Finally, by Corollary 14.2 we obtain the result.  $\square$

Next we need to extend Lemma 13.3 to collect Specht modules together. We call the sum  $c_1 + c_2 + \dots + c_j$  an *initial sum* of a composition  $\vec{c} = (c_1, c_2, \dots, c_k)$  for  $1 \leq j \leq k$ .

**Definition 16.3.** *For an interval  $[\vec{d}, \vec{b}]$  in the lattice of compositions  $\text{Comp}(n)$  let  $B^*(\vec{d}, \vec{b})$  be the skew-shape where the row lengths are given by  $d_1, d_2, \dots, d_{r-1}, d_r - 1$  and if the initial sum  $d_1 + \dots + d_j$  is equal to an initial sum of the composition  $\vec{b} - 1$ , then  $j$ th row and the  $(j + 1)$ st row overlap in one column. All other rows of  $B^*(\vec{d}, \vec{b})$  are non-overlapping.*


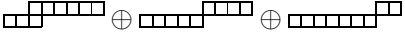

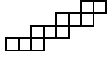
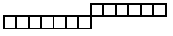
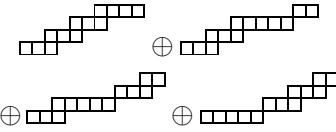
As an example, if  $\vec{d} = (2, 5, 4, 1, 3, 2)$  and  $\vec{b} = (2 + 5 + 4, 1 + 3, 2)$ , then  $B^*(\vec{d}, \vec{b})$  is the border strip with row lengths 2, 5, 4, 1, 3 and  $2 - 1 = 1$  which overlaps between the rows of length 4 and 1 and the rows of length 3 and 1. Note that  $B^*(\vec{d}, (n)) = A^*(\vec{d})$ .

The proof of the next lemma is the same as the proof of Lemma 13.3, that is, it uses jeu-de-taquin moves where two adjacent rows do not overlap.

**Lemma 16.4.** *Let  $\vec{b}$  and  $\vec{d}$  be two compositions in  $\text{Comp}(n)$  such that  $\vec{d} \leq \vec{b}$ . Then Specht module  $S^{B^*(\vec{d}, \vec{b})}$  is given by the direct sum*

$$S^{B^*(\vec{d}, \vec{b})} \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{d} \leq \vec{c} \leq \vec{b}} S^{B^*(\vec{c})}.$$

TABLE 1. The reduced homology groups of the order complex  $\Delta(\Pi_n^{\langle 3,5,7 \rangle} - \{\hat{1}\})$  for the even cases  $n = 8, 10, 12$  and  $14$ . Instead of writing out the notation  $S^{B^*(\vec{d}, \vec{b})}$  for the Specht modules, we have drawn the associated border shapes. Observe that when a row has three boxes, there is overlap with the row above.

$n$	$\tilde{H}_0$	$\tilde{H}_2$
8		0
10		0
12		
14		

In order to state the main result for the semigroup  $\Lambda = \langle a, a + d, a + 2d, \dots \rangle$  and the associated filter in the partition lattice, we need one last definition.

**Definition 16.5.** For a composition  $\vec{d}$  of  $n$  with entries in the set  $\{a\} \cup A$  let  $\vec{b}(\vec{d})$  be the composition greater than or equal to  $\vec{d}$  obtained by adding runs of entries of  $\vec{d}$  together where each run ends with the entry  $a$ .

As an example, for  $a = 3$ ,  $d = 2$  we have  $A = \{5, 7\}$ . Hence for the composition  $\vec{d} = (5, 3, 7, 5, 3, 3, 7, 5)$  we obtain  $\vec{b}(\vec{d}) = (5 + 3, 7 + 5 + 3, 3, 7 + 5) = (8, 15, 3, 12)$ .

**Remark 16.6.** Observe that the skew-shape  $S^{B^*(\vec{d}, \vec{b}(\vec{d}))}$  has the row lengths  $d_1, \dots, d_{r-1}, d_r - 1$  and satisfies the condition that  $d_i = a$  if and only if there is overlap between  $i$ th and  $(i + 1)$ st rows. See Definition 16.3.

**Theorem 16.7.** Let  $a$  and  $d$  be two positive integers and let  $\Pi_n^\Lambda$  be the filter in the partition lattice  $\Pi_n$  where each partition  $\pi$  consists of blocks whose cardinalities belong to the semigroup  $\Lambda$  generated by the arithmetic progression  $a, a + d, \dots, a + (a - 1) \cdot d$ . Then the  $i$ th reduced homology group of the order complex  $\Delta(\Pi_n^\Lambda - \{\hat{1}\})$  is given by the direct sum

$$\tilde{H}_i(\Delta(\Pi_n^\Lambda - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{d}} S^{B^*(\vec{d}, \vec{b}(\vec{d}))},$$

where the sum is over all compositions  $\vec{d}$  into  $i + 2$  parts such that every entry belongs to the set  $\{a\} \cup A = \{a, a + d, a + 2 \cdot d, a + (a - 1) \cdot d\}$ .

*Proof.* Let  $\vec{c}$  be a composition in the complex  $\Delta_n$ . Using the Morse matching given by Proposition 16.1 and Lemma 4.1, we obtain that a critical cell  $\vec{d}$  in the link  $\text{lk}_{\vec{c}}(\Delta_n) = \Delta_{c_1} * \Delta_{c_2} * \dots * \Delta_{c_k}$  is a composition  $\vec{d} \leq \vec{c}$  where the entries of  $\vec{d}$  belong to the set  $\{a\} \cup A$ . Furthermore, in the run of entries of  $\vec{d}$  that sums to the entry  $c_i$  of the composition  $\vec{c}$ , only the last entry of the run is allowed

TABLE 2. The reduced homology groups of the order complex  $\Delta(\Pi_n^{\langle 3,5,7 \rangle} - \{\hat{1}\})$  for the odd cases  $n = 9, 11, 13$  and  $15$ .

$n$	$\tilde{H}_1$	$\tilde{H}_3$
9		0
11		0
13		0
15		

to be equal to  $a$ . Using Theorem 11.6 we have

$$\begin{aligned} \tilde{H}_i(\Delta(\Pi_n^\Lambda - \{\hat{1}\})) &\cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in \Delta_n} \tilde{H}_{i-|\vec{c}|+1}(\text{lk}_{\vec{c}}(\Delta_n)) \otimes S^{B^*}(\vec{c}). \\ &\cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in \Delta_n} \bigoplus_{\vec{d}} S^{B^*}(\vec{c}), \end{aligned}$$

where the inner sum consists of critical compositions  $\vec{d}$  satisfying the conditions discussed in the previous paragraph and with  $|\vec{d}| = i + 2$ . By changing the order of summation we obtain

$$\tilde{H}_i(\Delta(\Pi_n^\Lambda - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{d}} \bigoplus_{\vec{c}} S^{B^*}(\vec{c}),$$

where the outer direct sum is over all compositions  $\vec{d}$  of  $n$  into  $i + 2$  parts where each part is in the set  $\{a\} \cup A$  and the inner direct sum is over all compositions  $\vec{c}$  greater than  $\vec{d}$  obtained by adding runs of entries of  $\vec{d}$  where an entry equal to  $a$  can only be at the end of a run. The inner direct sum is hence given by the Specht module  $S^{B^*}(\vec{d}, \vec{b}(\vec{d}))$  by Remark 16.6 and Lemma 16.4, and therefore the result follows.  $\square$

**Corollary 16.8.** *The order complex  $\Delta(\Pi_n^\Lambda - \{\hat{1}\})$  only has non-vanishing reduced homology in dimension  $i$  when  $n \equiv (i + 2) \cdot a \pmod{d}$ .*

*Proof.* Since all entries in the set  $\{a\} \cup A$  are congruent to  $a$  modulo  $d$ , we have  $n = \sum_{j=1}^{i+2} d_j \equiv (i + 2) \cdot a \pmod{d}$ .  $\square$

In Tables 1 and 2 we have explicitly calculated the reduced homology groups for the order complex  $\Delta(\Pi_n^{\langle 3,5,7 \rangle} - \{\hat{1}\})$  for  $8 \leq n \leq 15$ , that is, the case  $a = 3$  and  $d = 2$ . In this case the previous corollary implies that the order complex only has non-vanishing homology in dimensions of the same parity as  $n$ .

**Example 16.9.** When the integer  $d$  divides the integer  $a$ , the homology groups of  $\Pi_n^\Lambda$  have been studied. In this case, the filter  $\Pi_n^\Lambda$  consists of all partitions where the block sizes are divisible by  $d$



and the block sizes are greater than or equal to  $a$ . This filter was studied by Browdy [4], and our Theorem 16.7 reduces to Browdy's result; see Corollary 5.3.3 in [4].

**Example 16.10.** The previous example is particularly nice when  $d = 1$ . The semigroup  $\Lambda$  is given by  $\Lambda = \{n \in \mathbb{P} : n \geq a\}$  and the filter  $\Pi_n^\Lambda$  consists of all partitions where  $1, 2, \dots, a-1$  are forbidden block sizes. In this case it follows by Billera and Meyers [1] that  $\Delta_n$  is non-pure shellable. Additionally, Björner and Wachs [2] gave an  $EL$ -labelling of  $\Pi_n^\Lambda \cup \{\hat{0}\}$ . This order complex was also considered by Sundaram in Example 4.4 in [17].

## 17. THE PARTITION FILTER $\Pi_n^{\langle a, b \rangle}$

Let  $a$  and  $b$  be two relatively prime integers greater than 1. Let  $\Pi_n^{\langle a, b \rangle}$  be the filter in  $\Pi_n$  generated by all partitions whose block sizes are all  $a$  or  $b$ . As an example,  $\Pi_n^{\langle 2, 3 \rangle}$  consists of all partitions in  $\Pi_n$  with no singleton blocks. The corresponding complex  $\Delta_n$  in  $\text{Comp}(n)$  consists of all compositions of  $n$  whose parts are contained in the set  $\langle a, b \rangle = \{i \cdot a + j \cdot b : a, b \in \mathbb{N}\}$ . When  $a = 2$  and  $b = 3$  the complex  $\Delta_n$  is known as the complex of sparse sets; see [6, 10].

Following Theorem 4.1 in [6], we define the set  $A = \{n \in \mathbb{P} : n \equiv 0, a, b \text{ or } a + b \pmod{ab}\}$  and the function  $h : A \rightarrow \mathbb{Z}_{\geq -1}$  as follows:

$$(17.1) \quad h(n) = \begin{cases} \frac{2n}{ab} - 2 & \text{if } n \equiv 0 \pmod{ab}, \\ \frac{2(n-a)}{ab} - 1 & \text{if } n \equiv a \pmod{ab}, \\ \frac{2(n-b)}{ab} - 1 & \text{if } n \equiv b \pmod{ab}, \\ \frac{2(n-a-b)}{ab} & \text{if } n \equiv a + b \pmod{ab}. \end{cases}$$

Then Theorem 4.1 in [6] states that  $\Delta_n$  is either homotopy equivalent to a sphere or is contractible, according to

$$\Delta_n \simeq \begin{cases} S^{h(n)} & \text{if } n \in A, \\ \text{point} & \text{otherwise.} \end{cases}$$

Using equation (16.1) we see that if  $\vec{c} \in \Delta_n$  has any part not in  $A$ , then  $\text{lk}_{\vec{c}}(\Delta_n)$  is contractible. If each part of  $\vec{c}$  is in  $A$ , then  $\text{lk}_{\vec{c}}(\Delta_n) \simeq S^{h(c_1)} * \dots * S^{h(c_k)} = S^{h(\vec{c})}$ , where we define  $h(\vec{c}) = k - 1 + \sum_{j=1}^k h(c_j)$  for compositions  $\vec{c}$  with all parts in  $A$ , since the join of an  $n$ -dimensional sphere and an  $m$ -dimensional sphere is an  $(n + m + 1)$ -dimensional sphere. Note that  $h(\vec{c})$  is undefined for all other compositions.

For a composition  $\vec{c} = (c_1, \dots, c_k)$  of  $n$  with all of its parts in  $A$ , let  $\dim(\vec{c})$  denote the dimension of the reduced homology of  $\Delta(\Pi_n^{\langle a, b \rangle} - \{\hat{1}\})$  to which the composition  $\vec{c}$  contributes. That is,  $\dim(\vec{c})$  is given by

$$(17.2) \quad \dim(\vec{c}) = h(\vec{c}) + k - 1 = \sum_{i=1}^k h(c_i) + 2k - 2.$$

We can apply Theorem 11.6 to obtain

**Theorem 17.1.** *Let  $2 \leq a < b$  with  $\gcd(a, b) = 1$ . Then the  $i$ th reduced homology group of  $\Delta(\Pi_n^{\langle a, b \rangle} - \{\hat{1}\})$  is given by the direct sum of Specht modules  $\bigoplus_{\vec{c} \in F_i} S^{B^*(\vec{c})}$ , where  $F_i$  is the collection of compositions  $\vec{c}$  of  $n$  where all the parts are in the set  $A$  with  $\dim(\vec{c}) = i$ .*

*Proof.* We directly have

$$\begin{aligned}
\tilde{H}_i(\Delta(\Pi_n^{(a,b)} - \{\hat{1}\})) &\cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in \Delta} \tilde{H}_{i_{\vec{c}}}(\text{lk}_{\vec{c}}(\Delta_n)) \otimes S^{B^*}(\vec{c}) \\
&\cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in F_i} \tilde{H}_{i_{\vec{c}}}(S^{h(\vec{c})}) \otimes S^{B^*}(\vec{c}) \\
&\cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in F_i} S^{B^*}(\vec{c}). \quad \square
\end{aligned}$$

We now describe the top and bottom reduced homology of the order complex  $\Delta(\Pi_n^{(a,b)} - \{\hat{1}\})$ . We begin with the top homology.

**Proposition 17.2.** *Let  $2 \leq a < b$  with  $\gcd(a, b) = 1$ . Let  $r$  be the unique integer such that  $0 \leq r < a$  and  $n \equiv rb \pmod{a}$ . Then the top homology of  $\Delta(\Pi_n^{(a,b)} - \{\hat{1}\})$ , which occurs in dimension  $(n - r(b - a))/a - 2$ , is given by the direct sum of Specht modules  $\bigoplus_{\vec{c} \in R} S^{B^*}(\vec{c})$ , where  $R$  is the collection of compositions  $\vec{c}$  of  $n$  where exactly  $r$  of the parts are equal to  $b$  or  $a + b$ , and the remaining parts are all equal to  $a$ .*

*Proof.* We present two procedures that will change a composition  $\vec{c}$  into another composition  $\vec{c}'$  such that the dimension of contribution from  $\vec{c}'$  is greater than the contribution of  $\vec{c}$ , that is,  $\dim(\vec{c}) < \dim(\vec{c}')$ . The compositions which we cannot improve with this procedure are those described in the statement of the proposition.

We now describe the first replacement procedure. If the composition  $\vec{c}$  has a part of the form

- (i)  $jab$ , replace it with  $jb$   $a$ 's,
- (ii)  $jab + a$ , replace it with  $(jb + 1)$   $a$ 's,
- (iii)  $jab + b$ , replace it with  $jb$   $a$ 's and one  $b$ ,
- (iv)  $jab + a + b$ , replace it with  $(jb + 1)$   $a$ 's and one  $b$ ,

to obtain a new composition  $\vec{c}'$ . We claim that  $\dim(\vec{c}') - \dim(\vec{c}) = (b - a) \cdot j$ . We check the computation in the case (iv), the other three cases are similar. The difference  $\dim(\vec{c}') - \dim(\vec{c})$  only depends on the parts affected and the number of them. Hence

$$\begin{aligned}
\dim(\vec{c}') - \dim(\vec{c}) &= [(jb + 1) \cdot h(a) + h(b) + 2(jb + 2)] - [h(jab + a + b) + 2] \\
&= [jb + 2] - [2j + 2] = (b - a) \cdot j > 0,
\end{aligned}$$

using that  $h(a) = h(b) = -1$  and  $h(jab + a + b) = 2j$ . Hence this procedure increases the dimension.

Iterating this procedure we obtain a new composition with all the parts of the form  $a$ ,  $b$  and  $a + b$ .

The second replacement procedure is as follows. Assume that there are  $a$  parts of the composition  $\vec{c}$  that are different from  $a$ . Assume that  $p$  of these parts are equal to  $a + b$ , and hence  $a - p$  of them are equal to  $b$ . Replace these  $a$  parts with  $b + p$  parts equal to  $a$  to obtain a new composition  $\vec{c}'$ .

$$\begin{aligned}
\dim(\vec{c}') - \dim(\vec{c}) &= [(b + p) \cdot h(a) + 2(b + p)] - [p \cdot h(a + b) + (a - p) \cdot h(b) + 2a] \\
&= [b + p] - [a + p] = b - a > 0.
\end{aligned}$$

Hence the new composition  $\vec{c}'$  contributes to a homology of dimension  $b - a > 0$  greater than the composition  $\vec{c}$  does.

Iterating the last procedure, we are left with a composition  $\vec{c}$  where the number of parts different from  $a$  is at most  $a - 1$ . By considering the equation  $c_1 + \dots + c_k = n$  modulo  $a$ , we obtain the number of parts different from  $a$  is given by the integer  $r$  from the statement of the proposition.

Additionally, switching between one part of  $a + b$  and the two parts  $a$  and  $b$  does not change the dimension of the contribution of the composition. Finally, we compute the contribution of the composition  $(\underbrace{a, \dots, a}_{(n-br)/a}, \underbrace{b, \dots, b}_r)$  to obtain the desired dimension.  $\square$

**Corollary 17.3.** *Let  $2 \leq a < b$  with  $\gcd(a, b) = 1$ . Assume that  $n$  is divisible by  $a$ . Then the top homology of  $\Delta(\Pi_n^{(a,b)} - \{\hat{1}\})$ , which occurs in dimension  $n/a - 2$ , is the Specht module  $S^{B^*(a, a, \dots, a)}$ .*

*Proof.* When  $a$  divides  $n$ , then the integer  $r$  of Proposition 17.2 is 0. Thus the only contribution to reduced homology in dimension  $n/a - 2$  is given by  $(a, a, \dots, a)$ .  $\square$

We now turn our attention to the bottom reduced homology.

**Proposition 17.4.** *Let  $3 \leq a < b$  with  $\gcd(a, b) = 1$ . Let  $r$  and  $s$  be the two unique integers such that*

$$n \equiv rb \pmod{a}, \quad 0 \leq r < a, \quad n \equiv sa \pmod{b} \quad \text{and} \quad 0 \leq s < b.$$

*Then the bottom reduced homology of  $\Delta(\Pi_n^{(a,b)} - \{\hat{1}\})$  occurs in dimension  $2 \cdot \frac{n-sa-rb}{ab} + r + s - 2$ , and is given by the direct sum of Specht modules  $S^{B^*(\vec{c})}$  over all compositions  $\vec{c}$  such that the number of parts of  $\vec{c}$  of the form  $j \cdot ab + a$  and  $j \cdot ab + a + b$  is  $s$  and the number of parts of the form  $j \cdot ab + b$  and  $j \cdot ab + a + b$  is  $r$ .*

*Proof.* Just as in Proposition 17.2, we will define replacement procedures, where our goal now is to decrease the dimension of the homology that our composition contributes to, rather than increase it, as was the case in Proposition 17.2.

The first procedure takes  $b$  parts of the composition  $\vec{c}$  of the form  $jab + a$  and  $jab + a + b$  and subtracts  $a$  from each of these  $b$  parts, and adjoins a new part  $ab$ . Notice that the resulting new composition  $\vec{c}'$  remains a composition of  $n$ . Observe that  $h(jab) = h(jab + a) - 1$ ,  $h(jab + b) = h(jab + a + b) - 1$ , and  $h(ab) = 0$ . Hence the dimension  $\vec{c}'$  contributes to is  $\dim(\vec{c}') = \sum_{i=1}^{k+1} h(c'_i) + 2(k+1) - 2 = \sum_{i=1}^k h(c_i) - b + 2(k+1) - 2 = \dim(\vec{c}) - b + 2 < \dim(\vec{c})$ .

There is one small caveat. In the procedure, replacing a part  $a$  with 0 we obtain a weak composition, that is, we can introduce zero entries. Note the natural extension of the function  $h$  satisfies  $h(0) = -2$ . Assume that  $\vec{c}'$  has a zero entry, say in its last entry, and let  $\vec{c}''$  be the (weak) composition with this last entry removed. Then we have that  $\dim(\vec{c}') = \sum_{i=1}^{k+1} h(c'_i) + 2(k+1) - 2 = \sum_{i=1}^k h(c''_i) + 2k - 2 = \dim(\vec{c}'')$ . Thus zero entries can be removed without changing the dimension.

The second procedure is symmetric to the first in the two parameters  $a$  and  $b$ . That is, it takes  $a$  parts of the composition  $\vec{c}$  of the form  $jab + b$  and  $jab + a + b$  and subtracts  $b$  from each of these  $a$  parts and adjoins a new part  $ab$ . Now we have  $\dim(\vec{c}') = \dim(\vec{c}) - a + 2 < \dim(\vec{c})$ , using the fact that  $a \geq 3$ .

Iterating these two procedures we obtain a composition which has at most  $b - 1$  parts of the form  $jab + a$  and  $jab + a + b$ , and at most  $a - 1$  parts of the form  $jab + b$  and  $jab + a + b$ . Hence this composition satisfies the condition of the statement of the proposition. Finally, one has to observe that all such composition contribute to the same dimension.  $\square$

**Corollary 17.5.** *Let  $3 \leq a < b$ ,  $\gcd(a, b) = 1$  and let  $n$  be divisible by  $ab$ . Then the bottom reduced homology of the order complex  $\Delta(\Pi_n^{(a,b)} - \{\hat{1}\})$  is given by the permutation module  $M^{B^\#(ab, \dots, ab, ab)} = M^{B(ab, \dots, ab, ab-1)}$ .*

*Proof.* We have  $r = s = 0$ . Hence the compositions only have parts of the form  $j \cdot ab$ . The result follows from Lemma 13.3.  $\square$

We end with a complete description in the case when  $a = 2$ .

**Proposition 17.6.** *Let  $b$  be odd and greater than or equal to 3. Then the  $i$ th reduced homology of  $\Delta(\Pi_n^{(2,b)} - \{\hat{1}\})$  is given by the direct sum of Specht modules  $S^{B^*(\vec{c})}$  over all compositions  $\vec{c}$  with all parts congruent to 0 or 2 modulo  $b$ , where exactly  $(b(i+2) - n)/(b-2)$  entries of  $\vec{c}$  are congruent to 2 modulo  $b$ . The bottom reduced homology occurs in dimension  $\lceil n/b \rceil - 2$ . Furthermore, when  $b$  divides  $n$  the bottom reduced homology is given by the permutation module  $M^{B^\#(b, \dots, b, b)} = M^{B(b, \dots, b, b-1)}$ .*

*Proof.* Since  $a = 2$  the expression for  $h(n)$  in equation (17.1) reduces to  $h(n) = \lceil n/b \rceil - 2$  and the set  $A$  reduces to  $\{n \in \mathbb{P} : n \equiv 0, 2 \pmod{b}\}$ . Let  $\vec{c}$  be a composition of  $n$  into  $k$  parts, where each part belongs to the set  $A$ . Furthermore, assume that  $\vec{c}$  has  $s$  entries congruent to 2 modulo  $b$ . The contribution of  $\vec{c}$  to the reduced homology of  $\Delta(\Pi_n^{(2,b)} - \{\hat{1}\})$ , given by equation (17.2), is in dimension

$$\dim(\vec{c}) = \sum_{i=1}^k h(c_i) + 2k - 2 = \sum_{i=1}^k \left\lceil \frac{c_i}{b} \right\rceil - 2 = \frac{\sum_{i=1}^k c_i + s \cdot (b-2)}{b} - 2 = \frac{n + s \cdot (b-2)}{b} - 2.$$

Solving for  $s$  in this equation yields the desired expression.

For real numbers  $x$  and  $y$  we have the inequality  $\lceil x \rceil + \lceil y \rceil \geq \lceil x + y \rceil$ . Hence we obtain the lower bound on the dimension of the homology:  $\dim(\vec{c}) = \sum_{i=1}^k \lceil \frac{c_i}{b} \rceil - 2 \geq \lceil \frac{n}{b} \rceil - 2$ . When  $b$  divides  $n$  the only way to obtain equality in the previous inequality is when all the parts of the composition are divisible by  $b$ . The bottom reduced homology group is then the direct sum over all compositions  $\vec{c}$  of  $n$  where each part is divisible by  $b$ , that is,  $(b, b, \dots, b) \geq^* \vec{c}$ . Hence we obtain the permutation module  $M^{B^\#(b, \dots, b, b)} = M^{B(b, \dots, b, b-1)}$  by Lemma 13.3.  $\square$

## 18. CONCLUDING REMARKS

Using Theorem 11.6 we have been able to classify the action of  $\mathfrak{S}_{n-1}$  on the top homology of  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  for any complex  $\Delta \subseteq \text{Comp}(n)$ . In the case when  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  is shellable, is there an  $EL$ -labelling of  $\Pi_\Delta^* \cup \{\hat{0}\}$  that realizes this shelling order?

Is there a way we can classify the  $\mathfrak{S}_n$ -action on the homology groups of  $\Delta(\Pi_\Delta^* - \{\hat{1}\})$  rather than the  $\mathfrak{S}_{n-1}$ -action? Browdy described the matrices representing the action of  $\mathfrak{S}_n$  on the cohomology groups of the filter with block sizes belonging to the arithmetic progression  $k \cdot d, (k+1) \cdot d, \dots$ ; see [4, Section 5.4].

The partition lattice is naturally associated with the symmetric group, that is, the Coxeter group of type  $A$ . Miller [11] has extended the results about the filter  $\Pi_\Delta^*$  to other root systems. Hence it is natural to ask if our results for the filter  $\Pi_\Delta^*$  can be extended to other root systems.

Is there a non-pure shelling of the Frobenius complex generated by  $a$  and  $b$ ? Alternatively, is there a Morse matching for this Frobenius complex such that all the critical cells are facets? While we do have this property for  $\Lambda$  defined by an arithmetic progression as in Section 16, unfortunately the general matching given in [6] does not have this property.

Lastly, all of our results are based upon  $\Delta$  being a filter in the composition lattice  $\text{Comp}(n)$ . What if we remove the filter constraint? That is, let  $\Omega$  be an arbitrary collection of compositions of  $n$  not containing the extreme composition  $(n)$ . Define  $Q_\Omega^*$  to be all ordered set partitions  $\sigma = (C_1, C_2, \dots, C_k)$  such that  $\text{type}(\sigma) \in \Omega$  and containing  $n$  in the last block  $C_k$ . Let  $\Pi_\Omega$  be the image of  $Q_\Omega^*$  under the forgetful map  $f$ . What can be said about the homology groups and the homotopy type of the order complex  $\Delta(\Pi_\Omega)$ ? We need to understand the topology of the links  $\text{lk}_{\vec{c}}(\Omega)$ , even though these links are not themselves simplicial complexes.

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